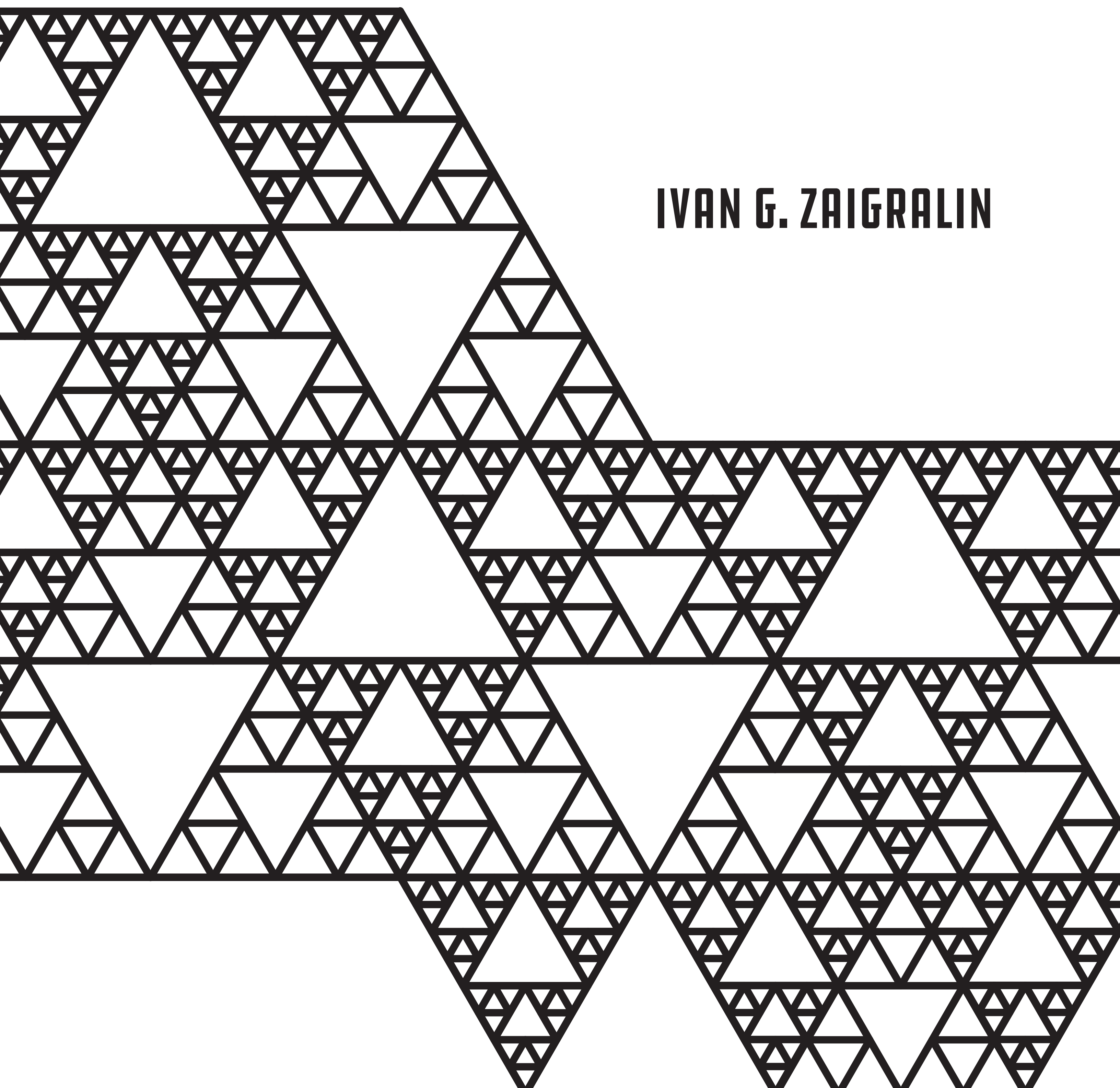
BASIC ALGEBRA

WITH APPLICATIONS



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CHAPTER 1

Concepts

1. Expressions, Relations, and Substitution

1.1. Expressions versus Relations.

DEFINITION 1.1.1. An *algebraic expression* or simply *expression* is a symbolic description of a single number. We call this number the *value* of the expression.

DEFINITION 1.1.2. Expressions are *equivalent* if they describe the same number, even though they may look very different. For example, expressions $2 \cdot 3$ and $5+1$ are equivalent, because they both evaluate to 6.

BASIC EXAMPLE 1.1.1. Here are some expressions:

expression	**what it stands for**
17	stands for the number 17
$10+7$	is a different expression which also stands for the number 17
$(3+5) \div 2$	stands for the number 4
x	is a variable expression, and it stands for an unknown number
$2x+y$	stands for a single unknown number; if we knew what expressions x and y stand for, we would be able to compute the value of $2x+y$

DEFINITION 1.1.3. An *equation* is a statement about the equality of two expressions:

$$A = B$$

An equation has an expression on the left, another expression on the right, and an equality sign in the middle. An equation is *true* if the value of the expression on the left is the same as the value of the expression on the right, and *false* otherwise.

It is important to distinguish between *equivalent expressions* and *equivalent equations*. When we say that *equations are equivalent*, it means that they have identical solutions (that is, the same numbers make them true when we substitute them for the variables), even though equations may look very different. The formal definition of equivalent equations is given later in the text.

BASIC EXAMPLE 1.1.2. Here are some equations:

equation	**truth value**
$4 = 4$	true
$17 = 10 + 7$	true
$3 = 1 + 1$	false
$x + 1 = 7$	true if the value of x is 6, false otherwise
$x = 6$	this one is equivalent to the one above: it's also true if the value of x is 6 and false otherwise

It is crucial to distinguish between expressions and equations. While intimately related to each other, they are completely different types of objects. For example, we can add the same number to both sides of any equation and obtain an equivalent equation, which is as true or false as the original. But if we add an arbitrary non-zero number to an expression, its value will change and we will end up with a different, unrelated expression.

DEFINITION 1.1.4. Informally speaking, every equation corresponds to a more general object we call a *binary relation*, or just *relation*, which is a true-or-false statement about two numbers. There are many different types of relations besides equations, and they all differ from *expressions* in one key way: expressions denote numbers, whereas relations denote truth or falsehood.

BASIC EXAMPLE 1.1.3. Besides equations, inequalities are some of the more useful relations for us:

relation	**truth value**
$4 < 4$	false, because 4 is not less than 4
$17 > 5 \cdot 2$	true, because 17 is greater than 10
$x > y$	true if the value of x is greater than the value of y, false otherwise

1.2. Algebraic Substitution. Algebraic substitution is the number one tool in our toolbox. It can be described as follows: any time and for any reason, we can replace any expression anywhere by any other expression which has the same value.

BASIC EXAMPLE 1.2.1. We know that expressions $6 \cdot 2$, $7+5$, and 12 all stand for the same number, so we can always swap one for the other in any expression, even if it is inside an equation. All of the following are perfectly equivalent statements:

$$\begin{aligned} 12x &= y+12 \\ (6\cdot 2)x &= y+12 \\ (6\cdot 2)x &= y+(7+5) \end{aligned}$$

As we substitute, it is never a bad idea to guard the value of the inserted expression with parentheses. With practice, we learn to omit the substitution parentheses in simple cases, like when we substitute a single positive integer, but more complicated expressions must be substituted with parentheses so that the correct order of operations is preserved.

THEOREM 1.2.1 (Algebraic Substitution). If the values of expressions A and B are the same, then we can *substitute* A by (B) in any expression, and obtain an equivalent expression.

EXAMPLE 1.2.1. Suppose that the value of x is -4. Substitute -4 for x in the expression $(3x-5)+x$.

ANSWER: $(3(-4)-5)+(-4)$

EXAMPLE 1.2.2. Suppose that the expression A has the same value as the expression $B+5$. Substitute $B+5$ for A in the equation

$$13A = 10 - A$$

SOLUTION: This example demonstrates why the parentheses are indispensable in some situations. Without the parentheses, we could make a mistake by writing something like $13\mathbf{B+5} = 10 - \mathbf{B+5}$, which would give us wrong values on both sides.

ANSWER: $13(B+5) = 10-(B+5)$

1.3. Common Types of Expressions. There are four basic types of expressions which are used heavily in this text: sums, products, terms, and factors.

DEFINITION 1.3.1. A *sum* is an algebraic expression consisting of expressions being added, and those expressions are called *terms*. Because subtraction can be thought of as adding a negative number, the terms of the sum can be separated by either pluses or minuses.

BASIC EXAMPLE 1.3.1. Here is a sum with 3 terms:

$$A + \mathbf{B} + C$$

Here is a sum with 4 terms, and its 3rd term -4 is negative:

$$a + \mathbf{x} - 4 + \mathbf{7y}$$

Here is a sum with 2 terms (everything inside the parentheses is a single term):

$$(x + y - 3z) + \mathbf{14}$$

Incidentally, the first term in the sum above is itself a sum with 3 terms:

$$x + \mathbf{y} - 3z$$

EXAMPLE 1.3.1. Identify all the terms in the expression

$$5y - 3x^2$$

ANSWER: 2 terms: $5y$, $-3x^2$

EXAMPLE 1.3.2. Identify all the terms in the expression

$$3(x + 1) + 5 + xy$$

SOLUTION: When identifying terms of an expression, we do not look inside the parentheses, so $3(x + 1)$ is a single term, even though another sum with two terms is locked up inside.

ANSWER: 3 terms: $3(x + 1)$, 5, xy

DEFINITION 1.3.2. A *product* is an algebraic expression consisting of expressions being multiplied, and those expressions are called *factors*.

BASIC EXAMPLE 1.3.2. Here is a product with 2 factors:

$$4\mathbf{x}$$

Here is a product with 3 factors, and its third factor is itself a sum with 2 terms:

$$A\mathbf{B}(X+7)$$

In general, sums and products can go into each other like the Russian dolls:

$$ABC - \mathbf{4(A+B)(X-Y)} + AB(X+7)$$

What we have above is a sum with 3 terms, and the second term $-4(\mathbf{A}+\mathbf{B})(X-Y)$ happens to be a product with three factors, and its second factor $A+\mathbf{B}$ happens to be a sum with two terms.

EXAMPLE 1.3.3. Identify all the factors in the expression: $\frac{2}{7}xy$

SOLUTION: Factors are easier to see if we make the multiplication visible:

$$\frac{2}{7} \cdot x \cdot y$$

We consider the rational number $2/7$ as a single factor.

ANSWER: 3 factors: $2/7$, x, y

EXAMPLE 1.3.4. Identify all the factors in the expression: $-10x(x+1)\frac{1}{z}$

SOLUTION: Unlike the addition operation, multiplication is often invisible, so it may be helpful to rewrite the expression with the multiplication dots shown:

$$(-10) \cdot x \cdot (x+1) \cdot \frac{1}{z}$$

ANSWER: 4 factors: -10, x, $x+1$, $\frac{1}{z}$

DEFINITION 1.3.3. A *numerical coefficient* or just *coefficient* is a multiplicative factor in a product. It usually looks like a numerical constant, but in complicated cases may be any expression. In the latter case, the variables appearing in the coefficient are often called parameters, and must be clearly distinguished from the other variables.

Traditionally, the coefficient is written on the left side of the product.

BASIC EXAMPLE 1.3.3.

product	**coefficient**
$3xy^2$	3
$-6a$	-6
$\frac{3}{4}x^3$	$\frac{3}{4}$
a^4b	1
$-x^2$	-1

Homework 1.1.

Determine the type and the meaning of the following formal statements. For equations and inequalities, determine whether they are true or false. For expressions, determine their numerical value.

1. $7 = 6 - 1$

2. $\frac{12}{4} = \frac{6}{2}$

3. $(14 + 2) - 9$

4. $(5 \cdot 3) + 4$

5. $14 < -1$

6. $5 \cdot 3 = 9 + 6$

7. $\frac{3 + 5}{4}$

8. $5 + 6 > 10$

Substitute the given expression for the variable. Use the parentheses and do **not** simplify the result.

9. Substitute 100 for x in the expression

$$2x + 1$$

10. Substitute 4 for x in the expression

$$5 - 4x$$

11. Substitute -7 for y in the expression

$$y + y^2$$

12. Substitute -12 for y in the expression

$$y^2(x + y)$$

13. Substitute $x + 1$ for k in the equation

$$k(k - 1) = 4k$$

14. Substitute $y - 2$ for h in the equation

$$\frac{2h}{3} = 5h$$

Identify all the terms in the given expression.

15. $5x - 6$

16. $1 + x + xy$

17. $-(x + 7) - 1 + x$

18. $(5x - 1) - (4 + x)$

19. $14xy^2$

20. $-(3 + 2x) - A$

21. $(1 + A) - (1 - 2A) + (4A + 3)$

22. $3(a + x) - 5(x - 3)(2y) + \frac{1}{7}$

Identify all the factors in the given expression.

23. $-6xy^5$

24. $14x^2y^2A^6$

25. $2x(u + 1)(u^2 - 1)$

26. $(A + B)^2(A - B)$

27. $a^2 - b$

28. $\frac{1}{4}(a + 1)\frac{1}{x}$

29. $\frac{1}{2}(ab)c$

30. $11(xy^2)(x + y)$

Homework 1.1 Answers.

1. A false equation

3. An expression with the value 7

5. A false inequality

7. An expression with the value 2

9. $2(100) + 1$

11. $(-7) + (-7)^2$

13. $(x + 1)((x + 1) - 1) = 4(x + 1)$

15. 2 terms: $5x$, -6

17. 3 terms: $-(x + 7)$, -1, x

19. 1 term: $14xy^2$

21. 3 terms: $1 + A$, $-(1 - 2A)$, $4A + 3$

23. 3 factors: -6, x, y^5

25. 4 factors: 2, x, $u + 1$, $(u^2 - 1)$

27. 1 factor: $a^2 - b$

29. 3 factors: $\frac{1}{2}$, ab, c

2. Real Number Axioms

2.1. Axioms. Axioms are the properties of numbers which are accepted without proof. In a sense, the axioms serve as a kind of definition for numbers, but unlike traditional definitions, which define new concepts in terms of old concepts, the axioms define the numbers and the arithmetic operations by telling us what they can and cannot do. The true nature of a number is never explained explicitly. We simply agree to call things numbers as long as they obey the axioms.

The first nine axioms appear almost too simple, and indeed they are. Almost anyone familiar with basic arithmetic knows these facts by heart.

AXIOM 2.1.1 (Closure of Arithmetic Operations). Adding any two real numbers results in a real number. Multiplying any two real numbers results in a real number.

BASIC EXAMPLE 2.1.1. Closure guarantees that addition and multiplication are defined for every pair of real numbers, and the result is a real number. When we substitute 5 for $2+3$, or 6 for $2 \cdot 3$, we can say that the substitution is justified by the closure.

AXIOM 2.1.2 (Commutativity of Addition). For any two real numbers X and Y,

$$X + Y = Y + X$$

AXIOM 2.1.3 (Associativity of Addition). For any two real numbers X and Y,

$$X + (Y + Z) = (X + Y) + Z$$

BASIC EXAMPLE 2.1.2. When considered together, commutativity and associativity of addition tell us that we can rearrange the terms of any sum, and perform the additions in any order we consider convenient. Because of that, when we work with expressions like

$$a + b + c + d$$

we are free to substitute them with equivalent expressions such as

$$d + c + b + a$$

or

$$(a + c) + (b + d)$$

AXIOM 2.1.4 (Commutativity of Multiplication). For any two real numbers X and Y,

$$XY = YX$$

AXIOM 2.1.5 (Associativity of Multiplication). For any two real numbers X and Y,

$$X(YZ) = (XY)Z$$

BASIC EXAMPLE 2.1.3. Just like with addition, commutativity and associativity of multiplication allow us to compute values of products by multiplying factors in any order, so a product with 4 factors such as

$$3 \cdot x \cdot y \cdot (a - 1)$$

is equivalent to any other product involving the same factors, regardless of the order of operations, for example

$$(a - 1) \cdot (3 \cdot y \cdot x)$$

or

$$(3 \cdot y) \cdot (x \cdot (a - 1))$$

AXIOM 2.1.6 (Additive Identity). There exists a real number 0, called *zero*, such that for all real numbers X,

$$X + 0 = X$$

AXIOM 2.1.7 (Multiplicative Identity). There exists a real number 1, called *one* or *unit*, such that for all real numbers X,

$$X \cdot 1 = X$$

While it may seem silly that we have to explicitly demand the existence of numbers such as zero and one, this is the natural way to proceed from the axioms. Note that other axioms apply to all numbers, but do not require the existence of any specific number. But once we have the identities and the closure of operations, we can immediately prove that there exists a number equal to $1 + 1$, and call it 2, and then also a number $2 + 1$, which we call 3, and so on.

AXIOM 2.1.8 (Additive Inverse). For each real number X, there exists a real number we call the *additive inverse of* X, or the *opposite of* X, written as $-X$, with the following property:

$$X + (-X) = 0$$

BASIC EXAMPLE 2.1.4. The opposite of 5 is -5 because $5 + (-5) = 0$.

At the same time, the opposite of -5 is 5 because $(-5) + 5 = 0$, so we can say 5 and -5 are opposites of each other. In general, it can be proven that every number has a unique opposite.

The opposite of zero is zero itself, because $0 + 0 = 0$.

AXIOM 2.1.9 (Multiplicative Inverse). For each non-zero real number X, there exists a real number we call the *multiplicative inverse of* X, or the *reciprocal of* X, written as $\frac{1}{X}$, $1/X$, or X^{-1}, with the following property:

$$X \cdot \left(\frac{1}{X}\right) = 1$$

Note that zero is very special. It is the only real number which is its own opposite, and the only real number with no reciprocal: there is no real number that would give us one when multiplied by zero.

BASIC EXAMPLE 2.1.5. The reciprocal of 2 is $1/2$, which can also be written as 0.5. To check, simply multiply them, and make sure to get 1 as the result. By the same token, the reciprocal of $1/2$ is 2.

In practice, obtaining the reciprocal amounts to writing the number as a fraction, and turning that fraction "upside-down". Recall that we can multiply fractions by multiplying numerators and multiplying denominators:

$$\frac{A}{B} \cdot \frac{X}{Y} = \frac{AX}{BY}$$

Recall also that we can write numbers like 2 as $2/1$. It should be easy to see now that the reciprocal of any fraction A/B is the fraction B/A, as long as neither A nor B is zero. Here are some more reciprocal pairs:

X	1	-1	4	$\frac{2}{3}$	$-\frac{1}{17}$	$-\frac{29}{31}$
$\frac{1}{X}$	1	-1	$\frac{1}{4}$	$\frac{3}{2}$	-17	$-\frac{31}{29}$

The last axiom deserves special attention. It is far less intuitive, and it takes a lot of practice to apply it correctly, or even to notice that it can be applied.

AXIOM 2.1.10 (Distributivity). For any three real numbers X, Y, and Z,

$$X(Y+Z) = XY + XZ$$

EXAMPLE 2.1.1. Use the distributivity to rewrite the expression without parentheses:

$$4(x+7)$$

SOLUTION:

$$\begin{aligned} 4(x+7) &= 4x+4\cdot 7 \\ &= 4x+28 \end{aligned}$$

ANSWER: $4x+28$

EXAMPLE 2.1.2. Use the distributivity to rewrite the expression without parentheses:

$$2(x+2y+3z)$$

SOLUTION: The distributive property can be shown to work for sums with more than two terms, so that each term gets multiplied by the number from the outside of the parentheses.

$$\begin{aligned} 2(x+2y+3z) &= 2\cdot x+2\cdot 2y+2\cdot 3z \\ &= 2x+4y+6z \end{aligned}$$

ANSWER: $2x+4y+6z$

EXAMPLE 2.1.3. Use the distributivity to rewrite the expression as a product with two factors:

$$4x+4$$

SOLUTION: The greatest common factor for these two terms is 4, so a single application of the distributive property gives a product of two factors: 4 and $x+1$.

$$\begin{aligned} 4x+4 &= 4x+4\cdot 1 && \text{multiplicative identity} \\ &= 4(x+1) && \text{distributivity} \end{aligned}$$

ANSWER: $4(x+1)$

EXAMPLE 2.1.4. Use the distributivity to rewrite the expression as a product with two factors:

$$ab + 3b$$

SOLUTION: The greatest common factor for these two terms is b, so a single application of the distributive property gives a product of two factors: $a + 3$ and b.

$$ab + 3b = (a + 3)b$$

ANSWER: $(a + 3)b$

2.2. Applying Axioms. Unlike the easy axioms, the distributivity of multiplication over addition will play a major role in our study, and we will refer to it explicitly throughout the text in order to justify algebraic substitutions. The rest of the axioms will still be in heavy use, but they will be applied under the hood, so to speak.

Before we leave this section, let us have some fun and see how the familiar algebraic concepts can be broken down into steps so tiny, that each step is justified by an axiom.

EXAMPLE 2.2.1. Simplify the expression by appealing to the axioms:

$$(x + 5) + (-x)$$

SOLUTION:

equivalent expressions	**axiom used for substitution**
$(x + 5) + (-x)$	
$(5 + x) + (-x)$	By commutativity of addition, $x + 5 = 5 + x$, so we were able to substitute $(5 + x)$ for $(x + 5)$ without a change in value.
$5 + (x + (-x))$	The whole expression was substituted by associativity of addition.
$5 + 0$	x and $-x$ are opposites and they must add up to zero, so we were able to substitute zero for $x + (-x)$.
5	zero is the additive identity, so we were able to substitute 5 for $5 + 0$.

ANSWER: 5

EXAMPLE 2.2.2. Rewrite the expression without parentheses and then simplify by appealing to the axioms:

$$3(2+x)+(-6)$$

SOLUTION: This is just like the previous example, but we will try to justify the steps in a concise manner, rather than describing them in with sentences. Since we are going strictly one substitution at a time, the substitutions we make should be obvious.

equivalent expressions	**axiom used for substitution**
$3(2+x)+(-6)$	
$(3\cdot 2+3x)+(-6)$	distributivity
$(6+3x)+(-6)$	closure
$(3x+6)+(-6)$	commutativity of +
$3x+(6+(-6))$	associativity of +
$3x+(0)$	additive inverse
$3x$	additive identity

ANSWER: $3x$

2.3. Inverses, Subtraction, Division.

THEOREM 2.3.1. Each number has a unique opposite, and zero is the only number which is its own opposite. An opposite of an opposite of a number is the number itself. Formally, for any real number x,

$$-(-x)=x$$

PROOF. We appeal directly to the axioms:

$$\begin{aligned}
-(-x) &= -(-x)+0 && \text{identity of } + \\
&= -(-x)+(-x+x) && \text{additive inverse} \\
&= (-(-x)+(-x))+x && \text{associativity of } + \\
&= 0+x && \text{additive inverse} \\
&= x && \text{identity of } +
\end{aligned}$$

□

THEOREM 2.3.2. Each non-zero number has a unique reciprocal. Only 1 and −1 are equal to their respective reciprocals:

$$1 = \frac{1}{1} \qquad -1 = \frac{1}{-1}$$

A reciprocal of a reciprocal of a number is the number itself. Formally, for any non-zero real number x,

$$\frac{1}{(1/x)} = x$$

PROOF. The proof is very similar to the previous one:

$$\begin{aligned} 1/(1/x) &= 1/(1/x) \cdot 1 && \text{identity of } \cdot \\ &= 1/(1/x) \cdot (1/x \cdot x) && \text{multiplicative inverse} \\ &= (1/(1/x) \cdot 1/x) \cdot x && \text{associativity of } \cdot \\ &= 1 \cdot x && \text{multiplicative inverse} \\ &= x && \text{identity of } \cdot \end{aligned}$$

□

Axioms do not mention subtraction or division, but we can define them in terms of addition, multiplication, and inverses.

DEFINITION 2.3.1. *Subtraction* is an operation defined in terms of the addition and the opposite. For any two real numbers X and Y,

$$X - Y = X + (-Y)$$

In other words, subtracting a number amounts to adding its opposite.

DEFINITION 2.3.2. *Division* is an operation defined in terms of the multiplication and the reciprocal. For any two real numbers X and Y,

$$X \div Y = X \cdot \frac{1}{Y}$$

In other words, dividing by a number amounts to multiplying by its reciprocal.

Other notations commonly used for $X \div Y$ are $\dfrac{X}{Y}$ and X/Y.

Now that we finally have all of the arithmetic operations, let us go back for a moment to the definitions of sum, product, term, and factor. Since subtraction is essentially addition, we are fully justified in calling the following expression a sum with 3 terms:

$$4x - 5 - Y$$

because it is equivalent to a sum of $4x$, the opposite of 5, and the opposite of Y

$$4x + (-5) + (-Y)$$

The same kind of reasoning applies to the division and the products. We can get away with calling this a product with 5 factors:

$$x \cdot y \cdot z \cdot \frac{1}{a} \cdot \frac{1}{b+c}$$

even though we usually write down an equivalent fraction:

$$\frac{xyz}{a(b+c)}$$

EXAMPLE 2.3.1 (Fraction Notation). Identify all the factors in the expression $\dfrac{x+y}{x-y}$

SOLUTION: The fraction notation is a division with a gotcha: unlike the regular division ÷ which behaves similarly to the multiplication · in the order of operations, the fraction bar *waits* for the values of its numerator and denominator, making it the very last operation to be performed within the fraction. One way to make sense of this is by enclosing numerator and denominator in parentheses, so that the order of operations becomes unambiguous:

$$\frac{x+y}{x-y} = \frac{(x+y)}{(x-y)}$$

It may be helpful to remember that every fraction comes equipped with these parentheses, but most of the time they are *invisible*, so to speak. Now we can see that this expression, when thought of as a product, has 2 factors: $(x+y)$ and $1/(x-y)$, which is the reciprocal of $(x-y)$.

ANSWER: 2 factors: $(x+y)$, $\dfrac{1}{(x-y)}$

2.4. Conclusion. Two concepts stand out in this section, and will be alluded to throughout the text. The distributive property is both indispensable and hard to grasp, and it is the only axiom we will keep calling by name. The way we defined subtraction and especially division will help us not to get lost while working with fractions and other complicated expressions.

The rest of the axioms are about to fade into shadows, but there is one last thing we want to stress, as it is the kind of insight that is simple yet profound at the same time. Not only the axioms *can* be used to break up everything we do into tiny, elementary, and fully justified steps; we also *must* be able to justify *everything we do* with axioms alone. Every single algebraic argument in this text is *justifiable* from the axioms and the logical principles, and nothing else would be recognized as valid. The rigorous study of the properties of numbers which arise directly from the axioms is called abstract algebra, and it is a fascinating, albeit a very difficult subject, consisting almost entirely of formal proofs, the simplest ones being similar to the step-by-step simplification examples above.

Homework 1.2.

Use the distributivity to rewrite the expression without parentheses:

1. $3(x+16)$

2. $4(x+6)$

3. $5(2+2a)$

4. $8(1+2y)$

5. $a(10+x)$

6. $b(x+y)$

7. $5(6x+9+4g)$

8. $6(2t+3+4s)$

Use the distributivity to rewrite the expression as a product with two factors:

9. $3a+3b$

10. $6y+6z$

11. $33x+11$

12. $21a+7$

13. $6x+12+18y$

14. $15a+45b+30$

15. $28p+7t+7$

16. $24b+b+ab$

Simplify the expression by appealing to the axioms:

17. $2(x+5)+3$

18. $5+3(4+a)$

19. $(6+x)+(-6)$

20. $g+(2+(-g))$

21. $x\left(7+\frac{1}{x}\right)+8$, given that $x \neq 0$

22. $12+a\left(4+\frac{1}{a}\right)$, given that $a \neq 0$

23. $5x+(6+3x)$

24. $(2a+9)+8a$

Homework 1.2 Answers.

1. $3x+48$

3. $10+10a$

5. $10a+ax$

7. $30x+45+20g$

9. $3(a+b)$

11. $11(3x+1)$

13. $6(x+2+3y)$

15. $7(4p+t+1)$

17.

expression	axiom
$2(x+5)$	
$(2\cdot x+2\cdot 5)+3$	distributivity
$(2x+10)+3$	closure
$2x+(10+3)$	associativity of +
$2x+13$	closure

19.

expression	axiom
$(6+x)+(-6)$	
$(x+6)+(-6)$	commutativity of +
$x+(6+(-6))$	associativity of +
$x+(0)$	additive inverse
x	additive identity

21.

expression	axiom
$x\left(7+\dfrac{1}{x}\right)+8$	
$\left(x\cdot 7+x\cdot\dfrac{1}{x}\right)+8$	distributivity
$(7x+(1))+8$	multiplicative inverse
$7x+(1+8)$	associativity of +
$7x+9$	closure

23.

expression	axiom
$5x+(6+3x)$	
$5x+(3x+6)$	commutativity of +
$(5x+3x)+6)$	associativity of +
$(5+3)x+6$	distributivity
$8x+6$	closure

3. The Number Line and Sets of Numbers

3.1. Roster.

DEFINITION 3.1.1. A *set* of numbers is any collection of numbers. Any number that belongs to the set is called an *element* or a *member* of that set.

DEFINITION 3.1.2 (Roster Notation). A set can be written down in the *roster notation*, which is a comma-separated list of its elements inside the curly brackets. A set with elements 1, 2, and 3 can be written as $\{1,2,3\}$. The order in which elements are listed is not important, so $\{1,2,3\}$ is the same set as $\{3,1,2\}$. A particular number either belongs to a set or it does not, just like a particular cat can be either alive or dead, so writing $\{7,7\}$ is just as redundant as saying "this cat is alive and alive".

BASIC EXAMPLE 3.1.1. Very large and infinite sets can be stated in roster notation using the ellipsis… For example, the infinite set of positive odd numbers can be stated as

$$\{1,\ 3,\ 5,\ 7,\ 9,\ 11,\ 13,\ \ldots\}$$

It is important to list enough elements for the pattern to become clear to the reader, and so the roster notation is too ambiguous for many interesting sets, like this one:

$$\{6,\ 28,\ 496,\ 8128,\ \ldots\}$$

What is the next number in this sequence? If you cannot tell, then perhaps the roster notation is not the best way to describe this set.

DEFINITION 3.1.3. A set with zero elements exists, and it is called the *empty set*. The traditional notation for the empty set is $\varnothing$, which is why it is probably not a good idea to strike through your zeroes: $\emptyset$. Roster notation $\{\}$ can also be used.

3.2. Integers.

DEFINITION 3.2.1. The set of *positive integers* is

$$\{1,\ 2,\ 3,\ 4,\ 5,\ \ldots\}$$

DEFINITION 3.2.2. The set of *non-negative integers* is

$$\{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots\}$$

which is exactly the same as the set of positive integers with one additional element: zero, which is neither positive nor negative. The standard notation for this set is $\mathbb{N}$, although some texts will use the same notation for positive integers.

Many texts use "whole numbers" and "natural numbers" to refer to either positive or non-negative integers, but we will not use either term in order to avoid confusion.

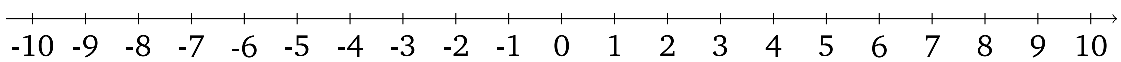
DEFINITION 3.2.3. The set of *integers* is

$$\{0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \ldots\}$$

which can also be written as

$$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

One great way to visualize the integers is the *number line*, which is a graphical representation of the collection of real numbers:



The number line extends infinitely far to the right and to the left, but we can only draw a finite segment of it. Every point on the number line corresponds to a real number, and the picture above highlights the integers.

The number line provides a convenient way to explain the addition of integers. Because addition is associative and commutative, the order in which numbers are added does not affect the result. To compute a sum, we start at zero on the number line. For each positive term of the sum we go that many units to the right on the number line, and for each negative term of the sum we go that many units to the left. The point where we end up is the value of the sum.

EXAMPLE 3.2.1. Find $5 + (-7)$

SOLUTION: 5 units to the right from zero take us to 5, and 7 units left take us to -2, so the value is -2.

ANSWER: -2

EXAMPLE 3.2.2. Find $-10 - (-7)$

SOLUTION: By definition of subtraction, this expression is equivalent to

$$-10 + (-(-7))$$

But the opposite of an opposite is the number itself: $-(-7) = 7$, so this is equivalent to

$$-10 + 7$$

Starting at zero, we move 10 units to the left to reach -10, and then 7 units to the right, and end up at -3.

ANSWER: -3

EXAMPLE 3.2.3. Find $-3+2-4-5$

SOLUTION: By definition of subtraction, this expression is equivalent to

$$(-3)+2+(-4)+(-5)$$

Starting from 0 on the number line,

- 3 units to the left take us to -3
- 2 units to the right take us to -1
- 4 units to the left take us to -5
- 5 units to the left take us to -10

ANSWER: -10

3.3. Set Builder.

DEFINITION 3.3.1 (Set-builder Notation). A set can be defined by the *set-builder notation*, which looks like this:

$$\left\{\text{expression} \,\middle|\, \text{relation}\right\}$$

Recall that relation is basically a statement which can be true or false. Defined as above, the set will contain values of all expressions for which the relation is true.

EXAMPLE 3.3.1. Write the set of even integers in set-builder notation.

SOLUTION: In roster notation, the set of even integers looks like this:

$$\{\ldots,-4,-2,0,2,4,6,8,\ldots\}$$

In set-builder, we can write

$$\left\{x \,\middle|\, x \text{ is even}\right\}$$

which reads "the set of numbers x such that x is even". This may look like an arcane way to rewrite a simple English statement, but the real usefulness of set-builder is made clear when we start using nontrivial expressions. For example, and with a lot more rigor, we can express the same set as

$$\left\{2k \,\middle|\, k \text{ is an integer}\right\}$$

This reads "the set of all values of the expression $2k$, where k is an integer". So this set has 2, since $2=2\cdot 1$; it has 4 since $4=2\cdot 2$, and so on. Defined this way, the set has every even integer, and none of the odd ones.

ANSWER: $\left\{2k \,\middle|\, k \text{ is an integer}\right\}$

EXAMPLE 3.3.2. List a few members of the set $\{3x \mid x \text{ is a negative integer}\}$

SOLUTION: Plug in a few negative integers into the expression to the left of the bar | to get elements such as $3(-1) = -3$, $3(-2) = -6$ and so on. State the answer in the roster notation.

ANSWER: $\{-3, -6, -9, -12, -15, \ldots\}$

EXAMPLE 3.3.3. List a few members of the set $\left\{ \frac{1}{k} \;\middle|\; k \text{ is a positive integer} \right\}$

SOLUTION: Plug in a few positive integers into the expression to the left of the | to get elements such as $1/1 = 1$, $1/2$, $1/3$ and so on. State the answer in the roster notation.

ANSWER: $\left\{1, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \frac{1}{5}, \ \ldots\right\}$

3.4. Rationals.

DEFINITION 3.4.1. The set of *rational numbers* is the collection of all numbers which can be represented by a fraction a/b where a is an integer and b is a positive integer:

$$\left\{ \frac{1}{2}, \ \frac{-1}{3}, \ \frac{17}{13}, \ \ldots \right\}$$

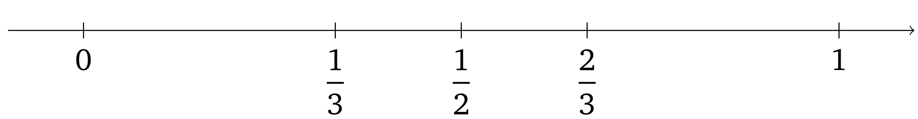
Entries above are listed in no particular order. It is possible, but not easy to come up with a pattern that would allow to list all rational numbers. The set-builder way of defining rationals completely removes the ambiguity:

$$\left\{ \frac{a}{b} \;\middle|\; a \text{ is an integer and } b \text{ is a positive integer} \right\}$$

The standard notation for the set of all rational numbers is $\mathbb{Q}$.

A number is rational as long as there is some way to write it as a fraction of integers. So zero is a rational number because $0 = 0/1$, and 0.5 is also a rational number because $0.5 = 1/2$.

Every rational number corresponds to a point on the number line. For example, $1/2$ is right in between 0 and 1, and $1/3$ is one third of the way from 0 to 1:



Rational numbers are what topologists call *dense* on the real line, meaning that infinitely many can be found in every little interval. However, not all points on the number line correspond to rational numbers.

3.5. Reals.

DEFINITION 3.5.1. The set of *real numbers* is the collection of all numbers with signs (positive or negative) and decimal representations. A decimal representation is a finite list of digits from 0 to 9 (the integer part), then a decimal point, then an infinite list of digits (the fractional part). It is traditional to omit an infinite tail of zeroes, and to avoid an infinite tail of nines:

$$2.100000\ldots = 2.1$$
$$17.99999\ldots = 18$$

The standard notation for the set of all real numbers is $\mathbb{R}$.

THEOREM 3.5.1. Each real number corresponds to a unique point on the real number line, and each point on the real number line corresponds to a unique real number.

All rational numbers are real. $1/2$ is a real number because it has a decimal representation 0.5, and $1/3$ is a real number because it has a decimal representation $0.333333\ldots = 0.\overline{3}$. A decimal representation of any integer fraction can be produced with long division. Some divisions (like $1/3$) seem to go on forever, but it can be shown that all rational numbers end in simple repeating patterns.

Rewriting decimal representations as fractions of integers is straightforward when possible.

EXAMPLE 3.5.1. Write the number 0.125 as a fraction of integers.

SOLUTION: The last digit 5 is in the third position after the decimal point, which corresponds to thousandths, so we can represent this number as a fraction with denominator 1000.

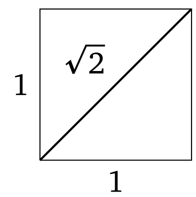
ANSWER: $\dfrac{125}{1000}$

EXAMPLE 3.5.2. Write the number 1075.56 as a fraction of integers.

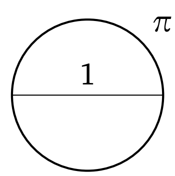
SOLUTION: This number ends with hundredths, so it is equivalent to

ANSWER: $\dfrac{107556}{100}$

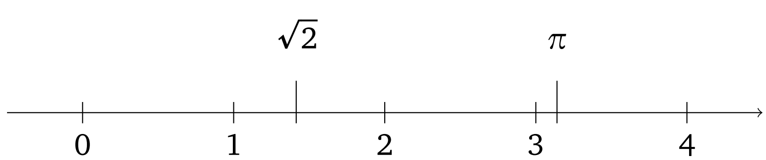
Probably the oldest known example of a real number that is not rational is $\sqrt{2}$, which is the length of the diagonal of a square with side of length 1. It can be written as $\sqrt{2} = 1.41421\ldots$, but unlike with rational numbers, the digits of the decimal representation of $\sqrt{2}$ do not have a simple repeating pattern.



Another famous example of a real number that is not rational is the ratio of a circle's circumference to its diameter, $\pi = 3.14159\ldots$, which is exactly the circumference (the length of the border) of a circle with diameter 1.



Real numbers are drawn on the number line in according to their decimal representations, so $\sqrt{2}$ is between 1 and 2 while π is shortly after 3:



DEFINITION 3.5.2. Real numbers which are not rational are also known as *irrational numbers*.

EXAMPLE 3.5.3. Identify all the integers in the set

$$\left\{-4, \quad 0, \quad \frac{1}{2}, \quad \frac{9}{3}, \quad 27.7\right\}$$

SOLUTION:

- -4 is an integer ✓
- 0 is an integer ✓
- $1/2$ is rational, but not an integer
- $9/3$ is equal to 3, so it is an integer ✓
- 27.7 is not an integer

We will state the answer as a subset consisting of integers, and note that we can put 3 instead of 9/3 because sets contain numbers, not the expressions which happen to define the numbers.

ANSWER: $\{-4,\ 0,\ 3\}$

EXAMPLE 3.5.4. Identify all the non-negative integers in the set

$$\left\{5, \quad 5.5, \quad \frac{5}{5}, \quad \frac{-10}{2}, \quad \frac{0}{-6}, \quad -12, \quad -0.17\right\}$$

SOLUTION:

- 5 is a non-negative integer ✓
- 5.5 is not an integer
- $5/5$ is equal to 1, which is a non-negative integer ✓
- $-10/2$ is equal to -5, which is not non-negative
- $0/(-6)$ is equal to 0, which is a non-negative integer ✓
- -12 is not non-negative
- -0.17 is neither non-negative nor an integer

ANSWER: $\{5,\ 1,\ 0\}$

3.6. Absolute Value.

DEFINITION 3.6.1. The *absolute value* of a real number x, written $|x|$, is the distance from the number to zero on the real number line. Alternatively, the absolute value of a non-negative number is the number itself, and the absolute value of a negative number is its opposite.

Absolute value bars alter the order of operations like parentheses, but the distributive property does not work on them. In general, we must know the value of the expression (that is, reduce it down to a single number) before removing the bars and the negative sign. It is a grave but common mistake to change pluses to minuses inside the absolute value bars before the value of the expression is known.

BASIC EXAMPLE 3.6.1.

$|-17| = 17$

$-|17| = -17$

$|4-13| = |-9| = 9$

$|2(-5)(-3)| = |30| = 30$

$2-|-5-3| = 2-|-8| = 2-8 = -6$

EXAMPLE 3.6.1. Find the value of the expression if $x = 5$

$$|x-7| + 2x$$

SOLUTION:

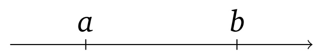
$$\begin{aligned} |x-7|+2x &= |(5)-7|+2(5) \\ &= |-2|+10 && \text{replaced } 5-7 \text{ by its value } -2 \\ &= 2+10 && \text{because } |-2| = 2 \\ &= 12 \end{aligned}$$

ANSWER: 12

3.7. Inequality Relations. The real number line provides an ordering of the set of real numbers. If the number line is oriented so that 1 is to the right of 0, then given any two distinct real numbers, the one on the right is greater than the one on the left.

DEFINITION 3.7.1. We will use five inequality relations in this text:

$a < b$ a is (strictly) less than b



$a > b$ a is (strictly) greater than b

$a \leq b$ a is less than or equal to b
$a \geq b$ a is greater than or equal to b
$a \neq b$ a is not equal to b

$<$ and $>$ are called *strict inequalities*, while $\leq$ and $\geq$ are called *non-strict inequalities*.

Other ways to read $a \leq b$ are "a is at most b" and "b is at least a".

EXAMPLE 3.7.1. Determine whether the inequality is true or false:

$$17 \geq 17$$

SOLUTION: $17 = 17$, so the non-strict inequality holds.

ANSWER: true

EXAMPLE 3.7.2. Determine whether the inequality is true or false:

$$-6 < -10$$

SOLUTION: -6 is to the right of -10 on the real number line, so it is greater than -10, and the inequality is false.

ANSWER: false

EXAMPLE 3.7.3. Determine whether the inequality is true or false:

$$3.1315 \geq 3.14$$

SOLUTION: The number 3.1315 is just a tad to the left of 3.1400 on the number line, so it is smaller, and the inequality fails.

ANSWER: false

EXAMPLE 3.7.4. Determine whether the inequality is true or false:

$$\frac{2}{6} < \frac{1}{3}$$

SOLUTION: To compare these fractions, we need to rewrite them with the same denominator somehow. If we multiply $1/3$ by $2/2$, for example, its value will not change, but the denominator will:

$$\frac{1}{3} = \frac{1}{3} \cdot \frac{2}{2} = \frac{1 \cdot 2}{3 \cdot 2} = \frac{2}{6}$$

So the inequality is equivalent to

$$\frac{2}{6} < \frac{2}{6}$$

which is false.

ANSWER: false

Homework 1.3.

Find the value of the expression:

1. $4-9$

2. $17-24$

3. $-2-6$

4. $-14-18$

5. $5-7-13$

6. $6+8-33$

7. $7-7$

8. $(-7)-(-5)$

9. $(-8)-(-5)-2$

10. $(-2)+(-5)+5$

11. $4-9-(-4)$

12. $(-12)-(-12)$

13. $8+(-6)+2$

14. $10+(-12)-3$

15. $-2+20-6-17$

16. $4+(-3)+(-2)+1$

17. Find the value of the expression

$$-12+x+3$$

if $x=-9$

18. Find the value of the expression

$$-10+14+y$$

if $y=-5$

19. Find the value of the expression

$$-4-z-1$$

if $z=-16$

20. Find the value of the expression

$$-k+4-10$$

if $k=-4$

List at least five elements of the given set using the roster notation:

21. $\left\{2k+100 \,\middle|\, k \text{ is a non-negative integer}\right\}$

22. $\left\{3(m-1) \,\middle|\, m \text{ is a negative integer}\right\}$

23. $\left\{\frac{x+1}{x} \,\middle|\, x \text{ is a positive integer}\right\}$

24. $\left\{x^2 \,\middle|\, x \text{ is a non-negative integer}\right\}$

---

25. Identify all the negative integers in the set

$$\left\{-7,\ 4,\ -7.5,\ -1,\ 0,\ \frac{-15}{3},\ 100\right\}$$

26. Identify all the non-negative integers in the set

$$\left\{-2,\ 0,\ \frac{16}{4},\ -100,\ \frac{1}{17}\right\}$$

27. Identify all the integers in the set

$$\left\{12,\ \frac{4}{3},\ -10.1,\ \frac{-12}{2},\ \pi\right\}$$

28. Identify all the positive integers in the set

$$\left\{\frac{4}{6},\ -4,\ \frac{60}{6},\ \frac{-10}{-10},\ \sqrt{2},\ \frac{0}{1}\right\}$$

Write the real number as a fraction of integers.

29. 0.4

30. 0.3

31. -0.17

32. -0.87

33. 3.1415

34. -2.718

Determine whether the inequality is true or false:

35. $7 \leq 7$

36. $9 > 5$

37. $-10 < -1$

38. $-5 \geq 4$

39. $0.1 < 0.1$

40. $-9 \geq -9$

41. $\frac{1}{8} > \frac{1}{4}$

42. $-6 \leq -10$

43. $\pi > 3$

44. $\sqrt{2} \leq 2$

Homework 1.3 Answers.

1. -5

3. -8

5. -15

7. 0

9. -5

11. -1

13. 4

15. -5

17. -18

19. 11

21. $\{100, 102, 104, 106, 108, \ldots\}$

23. $\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots\right\}$

25. $\{-7, -1, -5\}$

27. $\{12, -6\}$

29. $\frac{4}{10}$

31. $\frac{-17}{100}$

33. $\frac{31415}{10000}$

35. true

37. true

39. false

41. false

43. true

4. Properties of Real Numbers

4.1. Basic Properties. The following properties of real numbers are well-known, but we are going to highlight and prove them here, just to demonstrate the axiomatic approach we are taking. As you go through the proofs, you can have some fun by covering up the comments, and trying to name the axioms responsible for each step in the proof.

THEOREM 4.1.1. For any real number A,

$$0 \cdot A = 0$$

PROOF. We appeal directly to the axioms:

$$\begin{aligned} 0A &= 0A + 0 && \text{identity of } + \\ &= 0A + (1A - 1A) && \text{additive inverse} \\ &= (0A + 1A) - 1A && \text{associativity of } + \\ &= (0+1)A - 1A && \text{distributivity} \\ &= 1A - 1A && \text{identity of } + \\ &= 0 && \text{additive inverse} \end{aligned}$$

□

THEOREM 4.1.2. For all real numbers A and B,

$$(-A)B = A(-B) = -(AB)$$

and

$$(-A)(-B) = AB$$

As a consequence, multiplying numbers with opposite signs produces a negative result, while multiplying numbers with the same signs produces a positive result.

PROOF. To show that $(-A)B = -(AB)$, we appeal to the axioms, as well as the theorem above:

$$\begin{aligned} (-A)B &= (-A)B + 0 && \text{identity of } + \\ &= (-A)B + (AB - AB) && \text{additive inverse} \\ &= ((-A)B + AB) - AB && \text{associativity of } + \\ &= (-A + A)B - AB && \text{distributivity} \\ &= 0B - AB && \text{identity of } + \\ &= 0 - AB && \text{theorem 4.1.1} \\ &= -AB && \text{identity of } + \end{aligned}$$

Showing $A(-B) = -(AB)$ is very similar. To show the last identity, we appeal to the statements we've just proven:

$$\begin{aligned} (-A)(-B) &= -(A(-B)) \\ &= -(-(AB)) \\ &= AB && \text{theorem 2.3.1} \end{aligned}$$

□

EXAMPLE 4.1.1. Find $(-5)(-7)$

SOLUTION: $(-5)(-7) = 5 \cdot 7 = 35$

ANSWER: 35

EXAMPLE 4.1.2. Simplify $(-4)(7x)$

SOLUTION: $(-4)(7x) = -(4 \cdot 7x) = -28x$

ANSWER: $-28x$

EXAMPLE 4.1.3. Find $(-2)(-3)(-5)$

SOLUTION: $(-2)(-3)(-5) = (6)(-5) = -30$

ANSWER: -30

THEOREM 4.1.3. For any real number A,

$$(-1)A = -A$$

This is a striking result so many of us take for granted, telling us that taking an opposite of a number is the same as multiplying it by the opposite of the multiplicative identity.

PROOF. We appeal to the axioms, as well as the facts we have proven earlier:

$$\begin{aligned}
(-1)A &= (-1)A + 0 && \text{additive identity} \\
&= (-1)A + (1A - (1A)) && \text{additive inverse} \\
&= ((-1)A + 1A) - (1A) && \text{associativity of } + \\
&= ((-1) + 1)A - (1A) && \text{distributivity} \\
&= 0A - (1A) && \text{additive inverse} \\
&= 0 - (1A) && \text{theorem 4.1.1} \\
&= -(1A) && \text{additive identity} \\
&= -A && \text{multiplicative identity}
\end{aligned}$$

□

EXAMPLE 4.1.4. Remove the parentheses and simplify:

$$-(x - y + 2)$$

SOLUTION: We will rewrite the opposite as a multiplication by −1 and apply the distributive property. Recall also that subtracting a number is the same as adding its opposite.

$$\begin{aligned}
-(x - y + 2) &= (-1)(x + (-y) + 2) \\
&= (-1)x + (-1)(-y) + (-1)2 && \text{distributivity} \\
&= -x + y + (-2) && (-1)(-y) = 1y = y \\
&= -x + y - 2
\end{aligned}$$

ANSWER: $-x + y - 2$

Multiplying by −1 and then distributing may seem tedious, so in practice we simplify opposites of sums using an even nicer property.

THEOREM 4.1.4. For any collection of real numbers A, B, C, ..., the opposite of their sum is a sum of their opposites:

$$-(A + B + C + \ldots) = (-A) + (-B) + (-C) + \ldots$$

EXAMPLE 4.1.5. Remove the parentheses and simplify:

$$-(x - 3y + 5z - 1 - a)$$

SOLUTION: As we remove these parentheses, we replace each term of the sum by its opposite:

$$-(x-3y+5z-1-a)=-x+3y-5z+1+a$$

ANSWER: $-x+3y-5z+1+a$

EXAMPLE 4.1.6. Remove the parentheses and simplify:

$$-(-a+2-(5-b))$$

SOLUTION: We can remove the outer parentheses, and then the inner ones:

$$\begin{aligned} -(-a+2-(5-b)) &= a-2+(5-b) \\ &= a-2+5-b && -2+5=3 \\ &= a+3-b \end{aligned}$$

Alternatively, we can remove the inner parentheses first, but then of course we will get the same result:

$$\begin{aligned} -(-a+2-(5-b)) &= -(-a+2-5+b) \\ &= a-2+5-b \\ &= a+3-b \end{aligned}$$

ANSWER: $a+3-b$

4.2. Positive Integer Exponent.

DEFINITION 4.2.1. For any real number b and any positive integer n, the nth power of b is the product of n numbers b, written as

$$b^n = \underbrace{b\cdot b\cdot b\cdot\ldots\cdot b}_{n\text{ times}}$$

Within the expression b^n, b is called *base* and n is called *exponent*.

BASIC EXAMPLE 4.2.1. Base -6, exponent 2:

$$(-6)^2=(-6)(-6)=36$$

BASIC EXAMPLE 4.2.2. Base 5, exponent 3:

$$5^3=5\cdot5\cdot5=125$$

BASIC EXAMPLE 4.2.3.

$$(-3)^4 = (-3)(-3)(-3)(-3) = 81$$

BASIC EXAMPLE 4.2.4.

$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$$

Looking carefully at the examples above allows us to make a general statement about products of negative numbers:

THEOREM 4.2.1. A product of even number of negative factors is positive, while a product of odd number of of negative factors is negative.

EXAMPLE 4.2.1. Evaluate $(-1)^{1000}$

SOLUTION:

$$(-1)^{1000} = \underbrace{(-1)(-1)(-1)\dots(-1)}_{1000 \text{ times}}$$

This is too long to write out by definition, but since $(-1)(-1) = 1$ and the signs cancel in pairs, it is possible to prove that $(-1)^n = 1$ when n is even, and $(-1)^n = -1$ when n is odd.

ANSWER: 1

BASIC EXAMPLE 4.2.5. The following is an example of a notational convention: the exponent always supersedes the sign in the order of operations.

$$-7^2 = -(7^2) = -(7 \cdot 7) = -49$$

Compare it with

$$(-7)^2 = (-7)(-7) = 49$$

BASIC EXAMPLE 4.2.6. Unlike multiplication, the exponent does not distribute over addition. You can easily check that

$$(1+2)^2 = 3^3 = 9$$

is completely different from

$$1^2 + 2^2 = 1 + 4 = 5$$

Homework 1.4.

Simplify the expression:

1. $(-7)(-3)$

2. $(-1)(-17)$

3. $(-2a)(-4b)$

4. $(-6c)(-3d)$

5. $(-5)(-6)(-7)$

6. $(-10)(-3)(-1)$

Remove the parentheses and simplify:

7. $-(a-2b-6+c)$

8. $-(-4-3a+7b)$

9. $-(x+5)-(5-3y)$

10. $-(6-x)-(x+20)$

11. $-(10-(-4+2x))$

12. $-(-7-(19x-5))$

Find the value of the expression:

13. $(-4)^2$

14. $(-5)^2$

15. 2^5

16. $(-4)^3$

17. $(-7)^3$

18. $(-1)^6$

19. -3^4

20. $(0.5)^2$

21. $(-0.6)^2$

22. -0.4^2

23. $(-1)^{17}$

24. $(-1)^{54}$

Homework 1.4 Answers.

1. 21

3. $8ab$

5. -210

7. $-a+2b+6-c$

9. $-x-10+3y$

11. $-14+2x$

13. 16

15. 32

17. -343

19. -81

21. 0.36

23. -1

5. Order of Operations

5.1. Evaluating Sums and Products. The order of operations makes arithmetic expressions unambiguous. We end up evaluating from inside out, so to speak, starting with the innermost parentheses and absolute values.

Every expression in this text can be thought of as a sum of terms. Some of these terms will be products of factors, and some of these factors will have exponents. We think of subtraction as of adding the opposite, so our sums may have subtractions. We also think of division as of multiplying by the reciprocal, so our products may have divisions.

DEFINITION 5.1.1 (Order of Operations). To evaluate a sum of terms:

(1) Find the value of each term.
(2) Find the value of the sum by applying additions and subtractions from left to right.

Some of the terms will be products of several factors. To evaluate a product of factors with exponents:

(1) Find the value of each exponential expression by using a single factor on the left of the exponent as the base.
(2) Find the value of the product by applying multiplications and divisions from left to right.

At any point in this process we are free to remove or to insert parentheses by using the distributive property or any other axiom.

EXAMPLE 5.1.1. Evaluate the expression

$$5+4x-2x^2$$

if $x=3$

SOLUTION: This is a sum of 3 terms.

$$\begin{aligned} 5+4x-2x^2 &= 5+4(3)-2(3)^2 \\ &= 5+4(3)-2(9) && \text{evaluate the exponent} \\ &= 5+12-18 && \text{all terms are evaluated, time to add} \\ &= 17-18 \\ &= -1 \end{aligned}$$

ANSWER: -1

EXAMPLE 5.1.2. Evaluate the expression

$$(3-5)(-8+14)$$

SOLUTION: This is a product with 2 factors.

$$\begin{aligned} (3-5)(-8+14) &= (-2)(6) && \text{evaluate each factor} \\ &= -12 \end{aligned}$$

ANSWER: -12

EXAMPLE 5.1.3. Evaluate the expression

$$x^2 \div (x+8)(x+3)$$

if $x = -6$

SOLUTION: This is a product with 3 factors, the second factor being the reciprocal of $(x+8)$. Recall that multiplications and divisions are to be done left to right.

$$\begin{aligned} x^2 \div (x+8)(x+3) &= (-6)^2 \div ((-6)+8)((-6)+3) && \text{substitute } -6 \text{ for } x \\ &= (36) \div (2)(-3) && \text{all factors are evaluated} \\ &= (18)(-3) && 36/2 = 18 \\ &= -54 \end{aligned}$$

ANSWER: -54

EXAMPLE 5.1.4. Evaluate the expression

$$\frac{a-a^2}{5+a}$$

if $a = -3$

SOLUTION: It is important to remember that the fraction notation (Example 2.3.1) comes with invisible parentheses, and the division of numerator by denominator is last in the order of operations:

$$\frac{a-a^2}{5+a} = (a-a^2) \div (5+a)$$

So we can think of this fraction as of a product of 2 factors, and we have to find the value of each factor before we can divide:

$$\begin{aligned} \frac{a-a^2}{5+a} &= \frac{(-3)-(-3)^2}{5+(-3)} && \text{substitute } -3 \text{ for } a \\ &= \frac{-3-(9)}{2} \\ &= \frac{-12}{2} \\ &= -6 \end{aligned}$$

ANSWER: -6

EXAMPLE 5.1.5. Evaluate the expression

$$|-6\cdot 4| \div (-5+7)^3(-5^2)$$

SOLUTION: This can be seen as a product with 3 factors. We have to evaluate each factor before we can divide or multiply them. Note that we have to evaluate the base $(-5+7)$ before we can exponentiate it.

$$\begin{aligned} |-6\cdot 4| \div (-5+7)^3\cdot(-5^2) &= |-24| \div (2)^3\cdot(-25) \\ &= (24)\div 2^3\cdot(-25) \\ &= 24\div 8\cdot(-25) && \text{all factors are evaluated} \\ &= (3)(-25) \\ &= -75 \end{aligned}$$

ANSWER: -75

EXAMPLE 5.1.6. Evaluate the expression given that $x=2$ and $y=-1$

$$\frac{|x-y|}{-(x+y)^2}$$

SOLUTION: We view this fraction as a product of 2 factors:

$$\frac{|x-y|}{-(x+y)^2} = |x-y| \div (-(x+y)^2)$$

We have to evaluate the numerator and the denominator before we can divide.

$$\begin{aligned} \frac{|x-y|}{-(x+y)^2} &= \frac{|2-(-1)|}{-(2+(-1))^2} \\ &= \frac{|3|}{-(1)^2} \\ &= \frac{3}{-1} \\ &= -3 \end{aligned}$$

ANSWER: -3

5.2. Like Terms.

DEFINITION 5.2.1. *Like terms*, sometime called *similar terms*, are the terms which have all the same variable factors with the same corresponding exponents.

EXAMPLE 5.2.1. Simplify the expression $3xy-6xy$

SOLUTION: Both terms have the same variable factors xy, so we can factor them out using the distributivity:

$$\begin{aligned} 3xy-6xy &= (3-6)xy \\ &= (-3)xy \\ &= -3xy \end{aligned}$$

ANSWER: $-3xy$

EXAMPLE 5.2.2. Simplify the expression $5a-2(a^2-3a)$

SOLUTION: We begin by applying the distributive property to remove the parentheses:

$$\begin{aligned} 5a-2(a^2-3a) &= 5a-2a^2+2\cdot 3a \\ &= 5a-2a^2+6a \end{aligned}$$

Here $5a$ and $6a$ are like terms, while $-2a^2$ is different because it has exponent 2 instead of 1.

$$\begin{aligned} 5a - 2a^2 + 6a &= 5a + 6a - 2a^2 && \text{associativity of } + \\ &= (5+6)a - 2a^2 && \text{distributivity} \\ &= 11a - 2a^2 \end{aligned}$$

ANSWER: $11a - 2a^2$

EXAMPLE 5.2.3. Simplify the expression $5x - 1 - (3x - 1 + 10y)$

SOLUTION: We begin by removing parentheses, using the fact that the opposite of a sum is the sum of opposites. After that we use commutativity and associativity to change the order of addition and put like terms next to each other, and then simplify them using distributivity.

$$\begin{aligned} 5x - 1 - (3x - 1 + 10y) &= 5x - 1 - 3x + 1 - 10y \\ &= 5x - 3x + 1 - 1 - 10y && \text{properties of } + \\ &= (5-3)x + (1-1) - 10y && \text{distributivity} \\ &= 2x + 0 - 10y \\ &= 2x - 10y \end{aligned}$$

ANSWER: $2x - 10y$

Homework 1.5.

Evaluate the expression:

1. $3+8\div|-4|$

2. $8\div4\cdot2$

3. $(-6\div6)^3$

4. $5(-5+6)\cdot6^2$

5. $7-15\div3+6$

6. $(-9-(2-5))\div(-6)$

7. $|-7-5|\div(-2-2-(-6))$

8. $4-2\cdot|3^2-16|$

9. $\dfrac{-10-6}{(-2)^2}-5$

10. $\dfrac{2+4\cdot|7+2^2|}{4\cdot2+5\cdot3}$

11. $(6\cdot2+2-(-6))\left(-5+\left|\dfrac{-18}{6}\right|\right)$

12. $\dfrac{-13-2}{2-(-1)^3+(-6)-(-1-(-3))}$

13. $\dfrac{|3-(6+11)|}{7}+\dfrac{2^4}{(1-3)^2}$

14. $\dfrac{-5^2+(-5)^2}{|4^2-2^5|-2\cdot3}$

15. $\dfrac{-9\cdot2-(3-6)}{1-(-2+1)-(-3)}$

16. $\dfrac{5+3^2-24\div6\cdot2}{(5+3(2^2-5))+|2^2-5|^2}$

---

17. Evaluate the expression

$$-75\div x^2$$

if $x=-5$

18. Evaluate the expression

$$45\div3^2y(y-1)$$

if $y=3$

19. Evaluate the expression

$$6x\div12x^3$$

if $x=-2$

20. Evaluate the expression

$$-30\div a(a+4)^2$$

if $a=-6$

Simplify the expression by combining like terms:

21. $r-9+10$

22. $-4x+2-4$

23. $4b+6+1+7b$

24. $-7x-2-2x$

25. $-8px+5px$

26. $2r^2-r+6r^2$

27. $t-2t^2-t^2-7t$

28. $-5x+y+12x-y$

Simplify the expression by removing parentheses and combining like terms:

29. $9(b+10)+5b$

30. $4v - 7(1 - 8v)$

31. $-3(1 - 4x) - 4x$

32. $-8x + 9(-9x + 9)$

33. $-10 - 4(n - 5)$

34. $-6(5 - m) + 3m$

35. $(8n^2 - 3n) - (5 + 4n^2)$

36. $(7x^2 - 3) - (5x^2 + 6x)$

37. $a(x + 3) - 4ax + 2a$

38. $9x - 11xy - x(1 + 2y)$

39. $5(1 - 6k) + 10(k - 8)$

40. $-7(4x - 6) + 2(10x - 10)$

41. $b(m + y) - m(b - y)$

42. $(2b - 8) - 2(b^2 + 5)$

Homework 1.5 Answers.

1. 5

3. -1

5. 8

7. 6

9. -9

11. -40

13. 6

15. -3

17. -3

19. 8

21. $r+1$

23. $11b+7$

25. $-3px$

27. $-6t-3t^2$

29. $14b+90$

31. $-3+8x$

33. $10-4n$

35. $4n^2-3n-5$

37. $-3ax+5a$

39. $-20k-75$

41. $by+my$

6. Prime Numbers

6.1. Integer Division and Prime Numbers.

DEFINITION 6.1.1. A positive integer m is an *integer divisor* or just *divisor* of a positive integer n if dividing n by m leaves no remainder.

BASIC EXAMPLE 6.1.1.

- 4 is a divisor of 12 because $12/4 = 3$
- 5 is a divisor of 5 because $5/5 = 1$
- 3 is not a divisor of 5 because $5/3$ leaves a remainder of 2
- 2.5 is not a divisor of 5 because 2.5 is not an integer

DEFINITION 6.1.2. We call a positive integer *prime* when it has exactly two positive integer divisors: one and itself.

BASIC EXAMPLE 6.1.2.

- 1 is not prime because it only has a single divisor, itself
- 2 is prime because it has exactly two divisors: 1 and 2
- 3 is also prime
- 4 is not prime because it has three divisors: 1, 2, and 4
- 5 is prime
- 6 is not prime because it has four divisors: 1, 2, 3, and 6

THEOREM 6.1.1. The set of prime numbers is infinite, and it starts like this:

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, \dots\}$$

DEFINITION 6.1.3. A positive integer with three or more positive integer divisors is called *composite*. Incidentally, 1 is neither prime nor composite.

THEOREM 6.1.2 (Fundamental Theorem of Arithmetic). Every positive integer greater than 1 is either prime or is a product of prime factors, and this product is unique, up to the order of the factors.

EXAMPLE 6.1.1. Write 10 as a product of prime factors.

ANSWER: $2 \cdot 5$

EXAMPLE 6.1.2. Write 60 as a product of prime numbers.

SOLUTION: A quick and greedy way to success is to try to express the number as a product of smaller numbers, and keep doing that until nothing is left but primes.

$$60 = 6 \cdot 10 = (2 \cdot 3) \cdot (2 \cdot 5)$$

The order is not important within the product, but a traditional way to state the answer is by using the exponential notation, and listing lower bases first.

ANSWER: $2^2 \cdot 3 \cdot 5$

EXAMPLE 6.1.3. Write 231 as a product of prime numbers.

SOLUTION: When it is not immediately clear how to factor, we try dividing by every prime, starting with the lowest one, as many times as possible. $231/2$ leaves a remainder of 1, so 2 is not a factor. $231/3 = 77$, so 3 is a factor, and then

$$231 = 3 \cdot 77 = 3 \cdot 7 \cdot 11$$

ANSWER: $3 \cdot 7 \cdot 11$

EXAMPLE 6.1.4. Write 450 as a product of prime numbers.

SOLUTION:

$$450 = 10 \cdot 45 = (2 \cdot 5)(5 \cdot 9) = 2 \cdot 5 \cdot 5 \cdot (3 \cdot 3)$$

ANSWER: $2 \cdot 3^2 \cdot 5^2$

EXAMPLE 6.1.5. Write 448 as a product of prime numbers.

SOLUTION: Every even number has 2 as a factor, so we will keep dividing by 2 until we encounter an odd quotient:

$$\begin{aligned} 448 &= 2 \cdot 224 \\ &= 2 \cdot 2 \cdot 112 \\ &= 2 \cdot 2 \cdot 2 \cdot 56 \\ &= 2 \cdot 2 \cdot 2 \cdot (8 \cdot 7) \\ &= 2 \cdot 2 \cdot 2 \cdot (2 \cdot 2 \cdot 2) \cdot 7 \end{aligned}$$

ANSWER: $2^6 \cdot 7$

6.2. Fractions of Integers in Lowest Terms.

DEFINITION 6.2.1. A rational number is written in *lowest terms* if numerator and denominator have no common prime factors.

When numerator and denominator do have common factors, we can reduce the fraction to lowest terms by canceling them.

THEOREM 6.2.1. Common factors cancel in fractions. For any real number A and any two non-zero real numbers D and C,

$$\frac{AC}{DC} = \frac{A}{D}$$

EXAMPLE 6.2.1. Write 12/15 in lowest terms.

SOLUTION: If we can detect a common factor, we can simplify by canceling it.

$$\frac{12}{15} = \frac{3 \cdot 4}{3 \cdot 5} = \frac{4}{5}$$

ANSWER: $\frac{4}{5}$

EXAMPLE 6.2.2. Write 60/315 in lowest terms.

SOLUTION: Sometimes it is hard to identify common factors right away, but we can always completely factor both numerator and denominator, and then all of the common factors will become apparent.

$$\begin{aligned} 60 &= 6 \cdot 10 = (2 \cdot 3) \cdot (2 \cdot 5) = 2 \cdot 2 \cdot 3 \cdot 5 \\ 315 &= 5 \cdot 63 = 5 \cdot (7 \cdot 9) = 5 \cdot 7 \cdot (3 \cdot 3) = 3 \cdot 3 \cdot 5 \cdot 7 \end{aligned}$$

The common factors are 3 and 5:

$$\frac{60}{315} = \frac{2 \cdot 2 \cdot \mathbf{3} \cdot \mathbf{5}}{3 \cdot \mathbf{3} \cdot \mathbf{5} \cdot 7} = \frac{2 \cdot 2}{3 \cdot 7} = \frac{4}{21}$$

ANSWER: $\frac{4}{21}$

EXAMPLE 6.2.3. Simplify the fraction assuming all variables are non-zero:

$$\frac{4abx}{4by}$$

SOLUTION: Common factors 4 and b cancel.

ANSWER: $\frac{ax}{y}$

EXAMPLE 6.2.4. Simplify the fraction assuming all variables are non-zero:

$$\frac{6y^3}{18y^2}$$

SOLUTION:

$$\begin{aligned} \frac{6y^3}{18y^2} &= \frac{y^3}{3y^2} && \text{cancel common factor 6} \\ &= \frac{y}{3} && \text{cancel common factor } y^2 \end{aligned}$$

ANSWER: $\frac{y}{3}$

EXAMPLE 6.2.5. Simplify the fraction assuming all variables are non-zero:

$$\frac{60x^2}{40x}$$

SOLUTION:

$$\begin{aligned} \frac{60x^2}{40x} &= \frac{60 \cdot x}{40} && \text{cancel common variable factor } x \\ &= \frac{3 \cdot 20 \cdot x}{2 \cdot 20} && \text{20 is a common numerical factor} \\ &= \frac{3x}{2} && \text{rewrite in lowest terms} \end{aligned}$$

ANSWER: $\frac{3}{2}x$

Homework 1.6.

Write the number as a product of prime factors with exponents.

1. 12

2. 14

3. 16

4. 18

5. 19

6. 20

7. 22

8. 24

9. 26

10. 27

11. 30

12. 36

13. 40

14. 42

15. 45

16. 48

17. 160

18. 180

19. 625

20. 225

21. 196

22. 1000

Rewrite the rational number in lowest terms.

23. $\frac{21}{28}$

24. $\frac{21}{35}$

25. $\frac{16}{26}$

26. $\frac{12}{48}$

27. $\frac{12}{40}$

28. $\frac{75}{80}$

29. $\frac{52}{13}$

30. $\frac{110}{5}$

31. $\frac{4}{19}$

32. $\frac{46}{48}$

33. $\frac{150}{450}$

34. $\frac{72}{45}$

35. $\frac{27}{150}$

36. $\frac{8}{100}$

37. $\frac{24}{64}$

38. $\dfrac{25}{65}$

39. $\dfrac{90}{225}$

40. $\dfrac{253}{11}$

41. $\dfrac{160}{180}$

42. $\dfrac{625}{225}$

43. $\dfrac{196}{1000}$

44. $\dfrac{230}{460}$

Simplify the fraction assuming all variables are non-zero:

45. $\dfrac{18x^3}{6x}$

46. $\dfrac{7u^2}{28u}$

47. $\dfrac{-6ab^2}{-10b}$

48. $\dfrac{-8xy}{12xy}$

49. $\dfrac{-x}{5x^3}$

50. $\dfrac{20}{-60b}$

51. $\dfrac{48xy^2}{30x^2y}$

52. $\dfrac{46x^2y^2}{-26x^2}$

Homework 1.6 Answers.

1. $2^2 \cdot 3$

3. 2^4

5. 19

7. $2 \cdot 11$

9. $2 \cdot 13$

11. $2 \cdot 3 \cdot 5$

13. $2^3 \cdot 5$

15. $3^2 \cdot 5$

17. $2^5 \cdot 5$

19. 5^4

21. $2^2 \cdot 7^2$

23. $\frac{3}{4}$

25. $\frac{8}{13}$

27. $\frac{3}{10}$

29. 4

31. $\frac{4}{19}$

33. $\frac{1}{3}$

35. $\frac{9}{50}$

37. $\frac{3}{8}$

39. $\frac{2}{5}$

41. $\frac{8}{9}$

43. $\frac{49}{250}$

45. $3x^2$

47. $\frac{3ab}{5}$

49. $\frac{-1}{5x^2}$

51. $\frac{8y}{5x}$

7. Fractions and Rationals

7.1. Fraction Notation. Recall that for any non-zero real number B there is a reciprocal, which we denote by $1/B$, such that $B \cdot (1/B) = 1$. Recall also that division is multiplication by the reciprocal:

$$A \div B = \frac{A}{B} = A \cdot \frac{1}{B}$$

THEOREM 7.1.1. For any two non-zero real numbers A and B, the reciprocal of a product is the product of reciprocals:

$$\frac{1}{AB} = \frac{1}{A} \cdot \frac{1}{B}$$

BASIC EXAMPLE 7.1.1. This theorem generalizes to larger products as well:

$$\frac{1}{7} \cdot \frac{1}{10} \cdot \frac{1}{-2} = \frac{1}{7 \cdot 10 \cdot (-2)} = \frac{1}{-140}$$

This also also gives us a justification for multiplying fractions the way we do, by computing the product of numerators over the product of denominators:

$$\begin{aligned}
\frac{3}{7} \cdot \frac{2}{5} &= \left(3 \cdot \frac{1}{7}\right) \cdot \left(2 \cdot \frac{1}{5}\right) && \text{by definition of } \div \\
&= (3 \cdot 2) \cdot \left(\frac{1}{7} \cdot \frac{1}{5}\right) && \text{reorder multiplication} \\
&= (3 \cdot 2) \cdot \left(\frac{1}{7 \cdot 5}\right) && \text{apply theorem} \\
&= 6 \cdot \frac{1}{35} && \text{simplify} \\
&= \frac{6}{35} && \text{by definition of } \div
\end{aligned}$$

EXAMPLE 7.1.1. Identify all the factors in the expression by rewriting the fraction as a product:

$$\frac{3b}{xy}$$

SOLUTION: Formally, this fraction is a product of $(3b)$ and the reciprocal of (xy), which we can now separate into individual factors using the axioms and the basic properties:

$$\frac{3b}{xy} = (3b) \cdot \frac{1}{(xy)} = 3 \cdot b \cdot \frac{1}{x} \cdot \frac{1}{y}$$

ANSWER: 4 factors: $3 \cdot x \cdot \frac{1}{x} \cdot \frac{1}{y}$

EXAMPLE 7.1.2. Identify all the factors in the expression by rewriting the fraction as a product:

$$\frac{-5A(B-1)}{3X}$$

SOLUTION: It may be useful to rewrite the expression with multiplications made visible so that factors stand out more:

$$\frac{(-5) \cdot A \cdot (B-1)}{3 \cdot X}$$

ANSWER: 5 factors: $(-5) \cdot A \cdot (B-1) \cdot \frac{1}{3} \cdot \frac{1}{X}$

THEOREM 7.1.2. To multiply fractions, multiply numerators to obtain the new numerator, and multiply denominators to obtain the new denominator. For any two real numbers A and B and any two non-zero real numbers C and D,

$$\frac{A}{C} \cdot \frac{B}{D} = \frac{AB}{CD}$$

EXAMPLE 7.1.3. Multiply fractions:

$$\frac{x}{3} \cdot \frac{4}{5}$$

SOLUTION:

$$\frac{x}{3} \cdot \frac{4}{5} = \frac{x \cdot 4}{3 \cdot 5} = \frac{4x}{15}$$

ANSWER: $\frac{4x}{15}$

EXAMPLE 7.1.4. Multiply and simplify the result: $8 \cdot \frac{7}{12}$

SOLUTION: When multiplying (or dividing) a fraction by a number, it may be useful to represent that number as a fraction first:

$$\begin{aligned} 8 \cdot \frac{7}{12} &= \frac{8}{1} \cdot \frac{7}{12} \\ &= \frac{8 \cdot 7}{1 \cdot 12} \\ &= \frac{4 \cdot 2 \cdot 7}{4 \cdot 3} && \text{4 is a common factor} \\ &= \frac{2 \cdot 7}{3} \\ &= \frac{14}{3} \end{aligned}$$

ANSWER: $\frac{14}{3}$

EXAMPLE 7.1.5. Multiply and state the answer as a single fraction:

$$\frac{3}{2} \cdot \frac{x+1}{5}$$

SOLUTION: Recall that the fraction notation (Example 2.3.1) comes with invisible parentheses, and so the sum $x+1$ in the numerator becomes a single factor when we multiply numerators:

$$\begin{aligned} \frac{3}{2} \cdot \frac{x+1}{5} &= \frac{3}{2} \cdot \frac{(x+1)}{5} \\ &= \frac{3(x+1)}{2 \cdot 5} \\ &= \frac{3(x+1)}{10} \end{aligned}$$

ANSWER: $\frac{3(x+1)}{10}$

EXAMPLE 7.1.6. Find the area of a rectangular piece of fabric 5/4 feet long and 2/7 feet wide.

SOLUTION: The area of a rectangle is the product of its length and width:

$$\begin{aligned}\frac{5}{4}\cdot\frac{2}{7} &= \frac{5\cdot 2}{4\cdot 7} && \text{multiply the dimensions}\\ &= \frac{5}{2\cdot 7} && \text{cancel common factor 2}\\ &= \frac{5}{14}\end{aligned}$$

ANSWER: $\frac{5}{14}$ square feet

THEOREM 7.1.3. To divide a fraction by a fraction, multiply it by the reciprocal of the divisor. For any real number A and any three non-zero real numbers B, C and D,

$$\frac{A}{C}\div\frac{B}{D}=\frac{A}{C}\cdot\frac{D}{B}=\frac{AD}{CB}$$

EXAMPLE 7.1.7. Divide fractions:

$$\frac{6}{5}\div\frac{2}{5}$$

SOLUTION:

$$\begin{aligned}\frac{6}{5}\div\frac{2}{5} &= \frac{6}{5}\cdot\frac{5}{2}\\ &= \frac{6\cdot 5}{5\cdot 2}\\ &= \frac{6}{2}\\ &= 3\end{aligned}$$

ANSWER: 3

EXAMPLE 7.1.8. Divide fractions $\frac{8}{a-1} \div \frac{a+b}{2}$ and state the answer as a single fraction.

SOLUTION: Make the fraction notation parentheses visible in order to multiply sums correctly:

$$\begin{aligned} \frac{8}{a-1} \div \frac{a+b}{2} &= \frac{8}{(a-1)} \div \frac{(a+b)}{2} && \text{fraction parentheses} \\ &= \frac{8}{(a-1)} \cdot \frac{2}{(a+b)} && \text{multiplication by reciprocal} \\ &= \frac{8 \cdot 2}{(a-1)(a+b)} \\ &= \frac{16}{(a-1)(a+b)} \end{aligned}$$

ANSWER: $\frac{16}{(a-1)(a+b)}$

7.2. Changing the Denominator. Before we start adding fractions, we need to figure out a way to change a denominator without affecting the value of a fraction.

EXAMPLE 7.2.1. Rewrite the fraction $7/10$ so that it has the denominator 40.

SOLUTION: The challenge is in keeping the value of the fraction the same, while changing its appearance. If we multiply a fraction by 1, its value will not change. So when we multiply it by $4/4$, both numerator and denominator change, but the value of the fraction stays the same:

$$\frac{7}{10} = \frac{7}{10} \cdot \frac{4}{4} = \frac{7 \cdot 4}{10 \cdot 4} = \frac{28}{40}$$

ANSWER: $\frac{28}{40}$

EXAMPLE 7.2.2. Rewrite the fraction $-11/3$ so that it has the denominator 18.

SOLUTION:

$$-\frac{11}{3} = -\frac{11}{3} \cdot \frac{6}{6} = -\frac{11 \cdot 6}{3 \cdot 6} = -\frac{66}{18}$$

ANSWER: $-\frac{66}{18}$

Homework 1.7.

Identify all the factors in the expression by rewriting the fraction as a product:

1. $\dfrac{3x}{5}$

2. $\dfrac{6}{7a}$

3. $\dfrac{x+2}{2x(1+a)}$

4. $\dfrac{4yz}{a+b}$

5. $\dfrac{-4x(x+3)}{ab(5+c)}$

6. $\dfrac{12u}{-5(x+1)(x+2)}$

Perform fraction multiplication or division, simplify the result, and state the answer as a single fraction:

7. $\dfrac{1}{3}\cdot\dfrac{2}{5}$

8. $\dfrac{1}{8}\cdot\dfrac{3}{8}$

9. $9\cdot\dfrac{-2}{9}$

10. $2\cdot\dfrac{5}{6}$

11. $\dfrac{6}{5}\cdot\dfrac{11}{8}$

12. $\dfrac{-2}{3}\cdot\dfrac{3}{4}$

13. $-2\div\dfrac{3}{4}$

14. $\dfrac{-12}{7}\div\dfrac{-9}{5}$

15. $\dfrac{14}{5}\div 2$

16. $\dfrac{-13}{8}\div\dfrac{-15}{8}$

17. $\dfrac{5}{3}\div\dfrac{5}{6}$

18. $\dfrac{-4}{5}\cdot\dfrac{-13}{8}$

19. $\dfrac{1}{10}\div\dfrac{3}{2}$

20. $\dfrac{5}{3}\cdot\dfrac{5}{3}$

---

21. Find the area of a square with the side length of 3/4 feet.

22. Find the area of a rectangular strip of land 21/5 meters long and 2/3 meters wide.

Rewrite the fraction so that it has the given denominator:

23. $\frac{2}{3}$ with denominator 12

24. $\frac{-1}{4}$ with denominator 8

25. $\frac{-3}{14}$ with denominator 28

26. $\frac{7}{5}$ with denominator 20

27. $\frac{13}{7}$ with denominator 21

28. $\frac{9}{2}$ with denominator 16

29. $\frac{-11}{6}$ with denominator 600

30. $\frac{-5}{9}$ with denominator 99

Homework 1.7 Answers.

1. 3 factors: $3 \cdot x \cdot \frac{1}{5}$

3. 4 factors: $(x+2) \cdot \frac{1}{2} \cdot \frac{1}{x} \cdot \frac{1}{(1+a)}$

5. 6 factors: $(-4) \cdot x \cdot (x+3) \cdot \frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{(5+c)}$

7. $\frac{2}{15}$

9. -2

11. $\frac{33}{20}$

13. $\frac{-8}{3}$

15. $\frac{7}{5}$

17. 2

19. $\frac{1}{15}$

21. 9/16 square feet

23. $\frac{8}{12}$

25. $\frac{-6}{28}$

27. $\frac{39}{21}$

29. $\frac{-1100}{600}$

8. Fraction Addition

8.1. Adding Fractions with a Common Denominator.

THEOREM 8.1.1. A sum of fractions with a common denominator is the sum of numerators over the same denominator. For all real numbers A, B, and $C \neq 0$

$$\frac{A}{C}+\frac{B}{C} = \frac{A+B}{C}$$

$$\frac{A}{C}-\frac{B}{C} = \frac{A-B}{C}$$

PROOF. Express fractions as products, and then apply the distributive property to the common denominator:

$$\begin{aligned}\frac{A}{C}+\frac{B}{C} &= A\cdot\frac{1}{C}+B\cdot\frac{1}{C}\\ &= (A+B)\frac{1}{C}\\ &= \frac{A+B}{C}\end{aligned}$$

The subtraction case is similar. □

EXAMPLE 8.1.1. Simplify the expression $\frac{3}{8}+\frac{21}{8}$

SOLUTION:

$$\frac{3}{8}+\frac{21}{8}=\frac{3+21}{8}=\frac{24}{8}=3$$

ANSWER: 3

EXAMPLE 8.1.2. Simplify the expression $\frac{4}{15}-\frac{9}{15}$

SOLUTION:

$$\frac{4}{15}-\frac{9}{15}=\frac{4-9}{15}=\frac{-5}{15}=\frac{-1}{3}$$

ANSWER: $-\frac{1}{3}$

8.2. Least Common Multiple.

DEFINITION 8.2.1 (Least Common Multiple). The *least common multiple* for a collection of positive integers a, b, c, ... (*LCM* for short) is the smallest positive integer divisible by each of the a, b, c, ... without a remainder. To find the LCM, we factor each integer completely, and then take the product with each prime factor to the highest degree we found.

BASIC EXAMPLE 8.2.1.

integers	**LCM**
4, 12	12
4, 6	12
8, 10	40
2, 3, 5	30
4, 6, 8	24

EXAMPLE 8.2.1. Find the LCM for 16 and 20.

SOLUTION: When it's too hard to guess the LCM, we can factor each integer to expose the prime factors:

$$\begin{aligned} 16 &= 2^4 \\ 20 &= 2^2 \cdot 5 \end{aligned}$$

The highest power of 2 is 4, and the highest power of 5 is 1, so the LCM is

$$2^4 \cdot 5 = 16 \cdot 5 = 80$$

ANSWER: 80

EXAMPLE 8.2.2. Find the LCM for 34 and 40.

SOLUTION: Factor each integer:

$$\begin{aligned} 34 &= 2 \cdot 17 \\ 40 &= 2^3 \cdot 5 \end{aligned}$$

So the LCM is $2^3 \cdot 5 \cdot 17 = 680$

ANSWER: 680

EXAMPLE 8.2.3. Find the LCM for 121, 66, 8.

SOLUTION: Factor each integer:

$$\begin{aligned} 121 &= 11^2 \\ 66 &= 2 \cdot 3 \cdot 11 \\ 8 &= 2^3 \end{aligned}$$

So the LCM is $2^3 \cdot 3 \cdot 11^2 = 2904$

ANSWER: 2904

8.3. Adding Fractions of Integers.

DEFINITION 8.3.1. The *lowest common denominator* for a collection of rational numbers (*LCD* for short) is the LCM of their denominators.

THEOREM 8.3.1. Fractions with different denominators can be added by rewriting each fraction in an equivalent form using the LCD, and then adding as usual.

EXAMPLE 8.3.1. Simplify and state as a fraction in lowest terms:

$$\frac{5}{2} + \frac{3}{4}$$

SOLUTION: The LCD is 4, so we rewrite the first fraction:

$$\frac{5}{2} = \frac{5}{2} \cdot \frac{2}{2} = \frac{10}{4}$$

and then we can add:

$$\begin{aligned} \frac{5}{2} + \frac{3}{4} &= \frac{10}{4} + \frac{3}{4} \\ &= \frac{10+3}{4} \\ &= \frac{13}{4} \end{aligned}$$

ANSWER: $\frac{13}{4}$

EXAMPLE 8.3.2. Simplify and state as a fraction in lowest terms:

$$\frac{5}{3} - \frac{2}{5}$$

SOLUTION: The LCD is 15, so we rewrite both fractions:

$$\frac{5}{3} = \frac{5}{3} \cdot \frac{5}{5} = \frac{25}{15}$$

$$\frac{2}{5} = \frac{2}{5} \cdot \frac{3}{3} = \frac{6}{15}$$

and then we can subtract:

$$\frac{5}{3} - \frac{2}{5} = \frac{25}{15} - \frac{6}{15} = \frac{25-6}{15} = \frac{19}{15}$$

ANSWER: $\frac{19}{15}$

8.4. Mixed Numbers.

DEFINITION 8.4.1. A rational number is written as a *proper fraction* when the absolute value of the numerator is less than the absolute value of the denominator. In other cases it is called an *improper fraction*.

BASIC EXAMPLE 8.4.1. Some proper fractions: 17/19, −1/2, 10/20.

Some improper fractions: 3/2, 7/7, −101/100.

DEFINITION 8.4.2. A fraction is expressed as a *mixed number* when it is a sum of an integer and a proper fraction. Traditionally, the plus sign is not shown. For example, the number 3.25, which is equal to

$$3 + \frac{1}{4}$$

looks like this in the mixed number notation:

$$3\tfrac{1}{4}$$

EXAMPLE 8.4.1. Rewrite 60/13 as a mixed number.

SOLUTION: When we divide 60 by 13, the integer quotient is 4 and the remainder is 8, so

$$\frac{60}{13} = 4 + \frac{8}{13}$$

ANSWER: $4\frac{8}{13}$

EXAMPLE 8.4.2. Rewrite $7\frac{4}{5}$ as an improper fraction.

SOLUTION:

$$7\tfrac{4}{5} = 7 + \frac{4}{5} = \frac{35}{5} + \frac{4}{5} = \frac{35+4}{5} = \frac{39}{5}$$

ANSWER: $\dfrac{39}{5}$

Homework 1.8.

Perform the summation and simplify.

1. $\frac{1}{8}+\frac{3}{8}$

2. $\frac{4}{15}+\frac{5}{15}$

3. $\frac{14}{5}-\frac{24}{5}$

4. $\frac{12}{7}-\frac{19}{7}$

5. $\frac{17}{6\pi}+\frac{10}{6\pi}$

6. $\frac{10}{3}-\frac{22}{3}$

Find the LCM for given integers.

7. 3, 6

8. 5, 15

9. 6, 9

10. 10, 25

11. 2, 9, 10

12. 3, 8, 15

13. 48, 84

14. 28, 42

Perform the operations, simplify, state the answer as a single fraction in lowest terms:

15. $2+\frac{15}{8}$

16. $\frac{3}{5}+\frac{5}{4}$

17. $\frac{2}{5}-\frac{5}{4}$

18. $-1-\frac{2}{3}$

19. $\frac{-5}{7}-\frac{15}{8}$

20. $\frac{3}{2}+\frac{9}{7}$

21. $\frac{5}{3}-\left(-\frac{11}{6}\right)$

22. $\frac{13}{18}-\frac{4}{9}$

23. $\frac{1}{2}-\frac{11}{6}$

24. $-1-\left(\frac{1}{3}\right)$

25. $\frac{11}{8}-\frac{1}{2}$

26. $-\frac{4}{3}+\left(-\frac{2}{15}\right)$

27. $-6-\frac{5}{3}$

28. $-\frac{15}{8}+\frac{5}{3}$

29. $\frac{3}{2}\left(\frac{7}{9}-\frac{1}{9}\right)$

30. $\frac{2}{3}\left(\frac{6}{5}+\frac{9}{5}\right)$

31. $\left(\frac{5}{9}+\frac{7}{3}\right)\div\frac{13}{6}$

32. $\left(\frac{7}{8}-\frac{11}{4}\right)\div\frac{5}{4}$

Simplify the expression by combining like terms:

33. $\frac{5}{9}x + \frac{2}{3}x$

34. $\frac{11}{2}a - 3a$

35. $\frac{5y}{7} - \frac{5y}{21}$

36. $\frac{13b}{15} + \frac{11b}{45}$

37. $5a - \frac{3}{4}\left(\frac{2}{3} - \frac{1}{3}a\right)$

38. $b + \frac{5}{3}\left(\frac{1}{6} + \frac{9}{10}b\right)$

39. $\frac{1}{2}(6a+5b) + \frac{3}{2}(4a-9b)$

40. $\frac{2}{3}(9x-4y) - \frac{1}{3}(3a-2y)$

41. Evaluate the expression

$$6a + \frac{3}{2}$$

if $a = -7/5$

42. Evaluate the expression

$$3b + \frac{4}{15}$$

if $b = -4/7$

43. Evaluate the expression

$$\frac{3}{5}\left(x + \frac{1}{x}\right)$$

if $x = 2/3$

44. Evaluate the expression

$$\frac{3}{10}\left(y - \frac{3}{4}\right) \cdot \frac{1}{y}$$

if $y = 3/2$

Rewrite the mixed number as an improper fraction:

45. $1\frac{2}{3}$

46. $2\frac{2}{7}$

47. $-10\frac{5}{11}$

48. $-9\frac{4}{9}$

Rewrite the fraction as a mixed number:

49. $\frac{22}{5}$

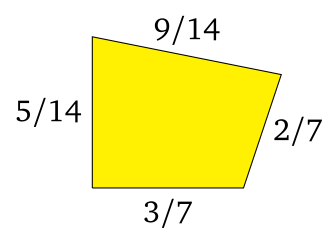
50. $\frac{50}{7}$

51. $-\frac{110}{21}$

52. $-\frac{82}{13}$

---

53. Find the perimeter of the shown shape if side lengths are given in inches:



54. Find the perimeter of a rectangular strip of land 21/5 meters long and 2/3 meters wide.

Homework 1.8 Answers.

1. $\frac{1}{2}$

3. -2

5. $\frac{9}{2\pi}$

7. 6

9. 18

11. 90

13. 336

15. $\frac{31}{8}$

17. $\frac{-17}{20}$

19. $\frac{-145}{56}$

21. $\frac{7}{2}$

23. $-\frac{4}{3}$

25. $\frac{7}{8}$

27. $-\frac{23}{3}$

29. 1

31. $\frac{4}{3}$

33. $\frac{11}{9}x$

35. $\frac{10y}{21}$

37. $\frac{21}{4}a-\frac{1}{2}$

39. $9a-11b$

41. $\frac{-69}{10}$

43. $\frac{13}{10}$

45. $\frac{5}{3}$

47. $-\frac{115}{11}$

49. $4\frac{2}{5}$

51. $-5\frac{5}{21}$

53. 12/7 inches

9. Translation

9.1. Expressions. The order of operations in mathematical expressions makes them unambiguous, and there are ways to carry some of that certainty into plain English. Here are some of the common ways to describe arithmetic expressions:

$a+b$	the sum of a and b
$a-b$	the difference of a and b
ab	the product of a and b
a/b	the quotient of a and b
a^2	the square of a

To see the difficulty with translation, consider the phrase

the sum of a and b times c

It could mean $a+bc$, or it could mean $(a+b)c$, and these expressions are not equivalent. Sometimes the ambiguity can be removed by finding a different phrasing:

the sum of a and the product of b and c

definitely means $a+bc$, while

the sum of a and b, times c

will probably be interpreted as $(a+b)c$, because of where the comma is placed in the sentence.

9.2. Equations. Many English sentences can be translated into equations almost word for word. Here are some of the most notorious and useful patterns:

BASIC EXAMPLE 9.2.1.

a	is	2	units	greater than	b
a	$=$		b	$+$	2

BASIC EXAMPLE 9.2.2.

a	is	5	units	less than	b
a	$=$		b	$-$	5

BASIC EXAMPLE 9.2.3.

a	is	6	times	greater than	b
a	$=$		6	$\cdot$	b

BASIC EXAMPLE 9.2.4.

a	is	7	times	less than		b
a	$=$			b	/	7

EXAMPLE 9.2.1. Describe the variables used to represent the given quantities, and write the statement as an algebraic relation:

One coffee costs as much as three donuts.

SOLUTION: Whenever it is not quite clear what to write, it may be useful to rephrase a statement to make it sound like one of the patterns above. The price of one coffee is three times the price of a donut. If c is the price of one coffee in dollars and d is the price of one coffee in dollars, then $c = 3d$. We will state the answer by defining the variables and their units, and then stating the relation between them.

ANSWER:
c is the price of one coffee in dollars
d is the price of one donut in dollars
$c = 3d$

EXAMPLE 9.2.2. Describe the variables used to represent the given quantities, and write the statement as an algebraic relation:

Chen is biking 2.5 *miles per hour faster than Keanu.*

SOLUTION: This statement compares speeds, so let c be Chen's speed, and let k be Keanu's speed, and then the sentence can be rephrased as

Chen's speed is 2.5 *miles per hour greater than Keanu's speed.*

ANSWER:
c is Chen's speed in mph
k is Keanu's speed in mph
$c = k + 2.5$

EXAMPLE 9.2.3. Describe the variables used to represent the given quantities, and write the statement as an algebraic relation:

The width of a FIBA regulation basketball court is 13 *meters shorter than its length.*

ANSWER:
w is the width of a court in meters
l is the length of a court in meters
$w = l - 13$

EXAMPLE 9.2.4. Describe the variables used to represent the given quantities, and write the statement as an algebraic relation:

A tiger weighs 3.1 *times less than a giraffe.*

ANSWER:
t is the weight of a tiger in kg
g is the weight of a giraffe in kg
$$t = \frac{g}{3.1}$$

9.3. Applications. We will solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations. As you construct your own solutions to applications, you may be using different variable names, and your equations may also be somewhat different, depending on how you translate English sentences. The final numerical answers, however, should always come out the same, regardless of translation.

EXAMPLE 9.3.1. A brownie recipe is asking for 350 grams of sugar, and a pound cake recipe requires 270 more grams of sugar than a brownie recipe. How much sugar is needed for the pound cake?

SOLUTION: Let b stand for the weight of sugar needed for the brownies, and p for the weight of sugar needed for the pound cake, both measured in grams. Then the phrase "pound cake recipe requires 270 more grams of sugar than a brownie recipe" translates as "p is 270 grams greater than b", or

$$p = b + 270$$

Since $b = 350$, we get

$$p = 350 + 270 = 620$$

ANSWER:
b and p are sugar amounts for brownies and pound cake respectively, in grams,
equation: $p = b + 270$
solution: $p = 620$

EXAMPLE 9.3.2. The size of a compressed file is 1.74 MiB, while the size of the original uncompressed file is 5.5 times greater. What is the size of the uncompressed file?

SOLUTION: We can restate the given informationas follows: the size of the uncompressed file is 5.5 times greater than the size of the compressed file:

$$u = 5.5c$$

ANSWER:
u and c are the sizes of the uncompressed and the compressed file respectively, in MiB,
equation: $u = 5.5c$
solution: $u = 9.57$

Homework 1.9.

Describe the variables used to represent the given quantities, and write the statements as algebraic relations.

1. Mateo is 5 years older than Carmen.

2. A buffalo is 4 times heavier than a tiger.

3. The radius of Jupiter is 11 times greater than the radius of the Earth.

4. A puppy is 6 pounds heavier than a kitten.

5. The air is 30 °F cooler than the water.

6. A donut is 2 dollars cheaper than a tea.

7. The highway is twice as wide as the alley.

8. In winter, the day is twice as short as the night.

9. Bahati is 7 inches shorter than Deion.

10. A bicycle is 4 times faster than a running person.

11. Jeff's pool is one and a half times longer than Kate's pool.

12. The postage this year is 2 cents more expensive than it was the last year.

13. The perimeter of a rectangle is 2 cm greater than 4 times the length of the shorter side.

14. The price of a nail is 1.5 dollars cheaper than one tenth of the price of a hammer.

15. The upload speed in bytes per second is 10 times slower than twice the download speed.

16. Min's wage in dollars per hour is 1.2 times greater than three times Yuze's wage.

Solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations.

17. Margo is 5 years old and Ivan is 34 years older than Margo. How old is Ivan?

18. The temperature in Boston is −3 °C and the temperature at the South Pole is 57 °C lower than the temperature in Boston. What is the South Pole temperature?

19. Akihiro runs 1.9 times faster than Motoko, who runs at 4.7 mph. How fast does Akihiro run?

20. In February 2017 the US House of Representatives had 362 congressmen. There were 286 fewer women in the House. How many congresswomen were there?

21. Tao's first car was 3 times cheaper than his second car, and he bought his second car for 7899 dollars. How much was Tao's first car?

22. The distance by car from Sacramento to San Francisco is 88 miles. The distance by car from Sacramento to Cool is 2.2 times shorter than the distance to San Francisco. How long is the trip from Sacramento to Cool?

Homework 1.9 Answers.

1. m is Mateo's age in years, c is Carmen's age in years, $m = c + 5$

3. J is the radius of Jupiter in meters, E is the radius of the Earth in meters, $J = 11E$

5. a is the temperature of the air in °F, w is the temperature of the water in °F, $a = w - 30$

7. h is the width of the highway in meters, a is the width of the alley in meters, $h = 2a$

9. b is Bahati's height in inches, d is Deion's height in inches, $b = d - 7$

11. J is the length of Jeff's pool in feet, K is the length of Kate's pool in feet, $J = 1.5K$

13. P is the perimeter of the rectangle in cm, x is the length of the shorter side in cm, $P = 4x + 2$

15. u is the upload speed in bps, d is the download speed in bps, $u = (2d)/10$

17.
M and I are Margo's and Ivan's ages respectively, in years,
equation: $I = M + 34$
solution: $I = 39$

19.
A and M are Akihiro's and Motoko's speeds respectively, in mph,
equation: $A = 1.9M$
solution: $A = 8.93$

21.
f and s are the prices of the first and the second cars respectively, in dollars,
equation: $f = s/3$
solution: $f = 2633$

Practice Test 1

1. Identify the terms in the expression

$$3+4x-5(x+y)$$

2. Identify the factors in the expression

$$-3x(y-1)(y+1)$$

3. Use the distributivity to rewrite the expression without parentheses:

$$-3(y-4x+1)$$

4. Use the distributivity to rewrite the expression as a product with 2 factors:

$$16x-12y-2$$

5. List at least four elements of the given set using the roster notation:

$$\left\{3-2x \;\middle|\; x \text{ is a negative integer}\right\}$$

6. Write the number 36 as a product of prime factors.

7. Rewrite the fraction 20/32 in lowest terms.

8. Simplify the expression by combining like terms:

$$2(8-y)-3(y-5)$$

9. Simplify the expression:

$$\frac{14-|5-7|}{30-3\cdot 2^3}$$

10. Evaluate the expression if $x=-4$

$$\frac{x-5}{1-2x}$$

11. Simplify the expression assuming that all variables are non-zero:

$$\frac{8x}{ya}\div\frac{6x}{5y}$$

12. Find the LCM for 10, 16.

13. Simplify the expression:

$$\frac{3}{4}+\frac{5}{6}\cdot\frac{3}{2}$$

14. Evaluate the expression if

$$a=-1 \quad \text{and} \quad b=7$$

$$\frac{a^5}{3}\div\left(4-\frac{b}{3}\right)$$

15. Simplify the expression by combining like terms:

$$-\left(3x-\frac{2}{5}y\right)+\frac{1}{5}(20x+7y)$$

16. *Describe the variables used to represent the given quantities, and write the statements as algebraic relations.*

The width of a rectangular room is 4 feet shorter than its length.

17. *Solve the application by describing the variables used to represent the quantities of interest, and stating appropriate equations.*

Arjun runs 1.1 times faster than Taras, who runs at 2.8 miles per hour. Find how fast Arjun runs.

Practice Test 1 Answers.

1. 3 terms: 3, $4x$, $-5(x+y)$

2. 4 factors: -3, x, $(y-1)$, $(y+1)$

3. $-3y+12x-3$

4. $2(8x-6y-1)$

5. $\{5,7,9,11,\ldots\}$

6. $2^2 \cdot 3^2$

7. $5/8$

8. $31-5y$

9. 2

10. -1

11. $\dfrac{20}{3a}$

12. 80

13. 2

14. $-1/5$

15. $x+\dfrac{9}{5}y$

16. w and l and width and length respectively, in feet.

$$w = l - 4$$

17. A and T are the respective speeds in miles per hour.

$$A = 1.1 \cdot T$$

$$A = 3.08$$

CHAPTER 2

Linear Equations

1. Properties of Equations

1.1. Equations and Solutions. Recall that an *equation* is a statement about equality of two expressions, and it has to be either true or false. When an equation has no variables, we can simplify the expressions and compare the numbers to find out whether the equations holds.

$$\begin{aligned} 3(4-5) &= 5-8 \\ 12-15 &= -3 \\ -3 &= -3 \end{aligned}$$

So the equation above was true all along.

When an equation contains a variable, it will typically be true for some values of the variable, and false for some others. Any value for which the equation is true will be known as its solution.

DEFINITION 1.1.1. A number is called a *solution* for the equation involving one variable if substituting this number for the variable makes the equation true. The set of all solutions for the equation is called its *solution set*. To *solve* the equation means to find its solution set.

EXAMPLE 1.1.1. Determine if -7 is a solution for the equation

$$3-x=5x$$

SOLUTION: Substitute (-7) for x everywhere in the equation and simplify both sides:

$$\begin{aligned} 3-x &= 5x \\ 3-(-7) &= 5(-7) \\ 3+7 &= -35 \\ 10 &= -35 \end{aligned}$$

The equation is false, so -7 is not a solution.

ANSWER: No

EXAMPLE 1.1.2. Determine if 6 is a solution for the equation

$$w-4=\frac{1}{3}w$$

SOLUTION: Substitute 6 for w everywhere in the equation and simplify both sides:

$$\begin{aligned} w-4 &= \frac{1}{3}w \\ (6)-4 &= \frac{1}{3}(6) \\ 2 &= 2 \end{aligned}$$

The equation is true, so 6 is a solution.

ANSWER: Yes

DEFINITION 1.1.2. Equations are *equivalent* if they have the same solutions sets.

We will typically solve equations by finding equivalent equations with known solution sets. For example, simplifying an expression cannot change its value, so any algebraic simplification produces an equivalent equation.

EXAMPLE 1.1.3. Solve the equation $x=2(3-5)$

SOLUTION: The following equations are equivalent:

$$\begin{aligned} x &= 2(3-5) \\ x &= 2(-2) \\ x &= -4 \end{aligned}$$

The last equation is true when x is -4, and false otherwise. We will state the answer as a set with one element, using the roster notation.

ANSWER: $\{-4\}$

1.2. Addition Property for Equations.

THEOREM 1.2.1. Adding the same number or expression to both sides of an equation produces an equivalent equation. Formally, for all expressions A, B, and C the following two equations are equivalent:

$$\begin{aligned} A &= B \\ A+C &= B+C \end{aligned}$$

Since every number has an opposite, and subtracting a number means adding its opposite, we can also subtract the same number on both sides of an equation, and obtain an equivalent equation:

$$A-C = B-C$$

EXAMPLE 1.2.1. Solve the equation $x-3=10$

SOLUTION: We try to isolate x on the left side by canceling -3:

$$\begin{aligned} x-3 &= 10 && \\ x-3+3 &= 10+3 && \text{added 3 to both sides} \\ x &= 13 && \text{combined like terms} \end{aligned}$$

ANSWER: $\{13\}$

EXAMPLE 1.2.2. Solve the equation $4x=3x$

SOLUTION: We don't know what number $3x$ stands for, but we know it is a number, and every number has an opposite, so we can add $-3x$ to both sides, which is the same as subtracting $3x$.

$$\begin{aligned} 4x &= 3x && \\ 4x-3x &= 3x-3x && \text{added } -3x \text{ to both sides} \\ x &= 0 && \text{combined like terms} \end{aligned}$$

ANSWER: $\{0\}$

1.3. Multiplication Property for Equations.

THEOREM 1.3.1. If C is an expression denoting a non-zero real number, then multiplying both sides of an equation by C produces an equivalent equation. Formally, for all expressions A and B, and all non-zero numbers C the following two equations are equivalent:

$$\begin{aligned} A &= B \\ AC &= BC \end{aligned}$$

Since every non-zero number has a reciprocal, and dividing by a number means multiplying by its reciprocal, we can also divide both sides by the same non-zero number, and obtain an equivalent equation:

$$\frac{A}{C} = \frac{B}{C}$$

Multiplying both sides of an equation by zero almost guarantees a non-equivalent result. For example, the equation $x = 3$ has a single solution, while the equation $0 \cdot x = 0 \cdot 3$ is true for every real number x, and so has infinitely many solutions.

EXAMPLE 1.3.1. Solve the equation $5x = 75$

SOLUTION: We would like to get rid of 5 on the left so that x is isolated, so we divide both sides by 5, which is the same as multiplying both sides by $1/5$:

$$\begin{aligned} 5x &= 75 \\ \frac{1}{5}(5x) &= \frac{1}{5}(75) \\ x &= 15 \end{aligned}$$

ANSWER: $\{15\}$

EXAMPLE 1.3.2. Solve the equation $-\frac{3}{7}x = \frac{6}{5}$

SOLUTION: We would like to isolate x on the left side, so we need to cancel the coefficient $-3/7$. Multiply both sides by $-7/3$, which is the reciprocal of $-3/7$:

$$\begin{aligned} -\frac{3}{7}x &= \frac{6}{5} \\ -\frac{7}{3}\left(-\frac{3}{7}x\right) &= -\frac{7}{3}\left(\frac{6}{5}\right) \end{aligned}$$

On the left, reciprocals cancel. On the right we reduce the fraction to lowest terms by canceling the common factor 3.

$$\begin{aligned} x &= -\frac{7\cdot(2\cdot 3)}{3\cdot 5} \\ x &= -\frac{14}{5} \end{aligned}$$

ANSWER: $\left\{-\frac{14}{5}\right\}$

EXAMPLE 1.3.3. Solve the equation $6x-4=20$

SOLUTION: Addition and multiplication properties may be exploited any number of times and in any order. Here we can isolate x on the left side if we add 4 first, and then divide both sides by 6.

$$\begin{aligned} 6x-4 &= 20 && \\ 6x-4+4 &= 20+4 && \text{addition property} \\ 6x &= 24 && \text{combined like terms} \\ \frac{6x}{6} &= \frac{24}{6} && \text{multiplication property} \\ x &= 4 && \end{aligned}$$

ANSWER: $\{4\}$

1.4. Applications. We will solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations. As you construct your own solutions to applications, you may be using different variable names, and your equations may also be somewhat different, depending on how you translate English sentences. The final numerical answers, however, should always come out the same, regardless of translation.

EXAMPLE 1.4.1. The profit of ACME Corporation in the second quarter was 8.84 billion dollars, which was 1.3 times greater than the profit in the first quarter. What was ACME's profit in the first quarter?

SOLUTION: Let f and s be the profits in the first and second quarter respectively, measured in billions of dollars. Then we can translate the statement above as "s is 1.3 times greater than f" or

$$s = 1.3f$$

Since $s = 8.84$, we can substitute it and solve the resulting equation for f:

$$\begin{aligned} 8.84 &= 1.3f \\ \frac{8.84}{1.3} &= \frac{1.3f}{1.3} \\ 6.8 &= f \end{aligned}$$

ANSWER:
f and s are the first and the second quarter profits respectively, in billions of dollars,
equation: $s = 1.3f$
solution: $f = 6.8$

EXAMPLE 1.4.2. The width of a rectangle is 125 meters shorter than the length. Find the length if the width is 205 meters.

SOLUTION: Let w and l be the width and the length respectively, measured in meters. Then the statement "The width of a rectangle is 125 meters shorter than the length" translates as "w is 125 meters less than l" or

$$w = l - 125$$

Since $w = 205$, we can substitute it and solve the resulting equation for l:

$$\begin{aligned} 205 &= l - 125 \\ 205 + 125 &= l - 125 + 125 \\ 330 &= l \end{aligned}$$

ANSWER:
w and l are the width and the length respectively, in meters,
equation: $w = l - 125$
solution: $l = 330$

Homework 2.1.

Determine whether the given set is a possible solution set for the given equation.

1. $\{4\}$, $5x+7=29$

2. $\{8\}$, $6-x=-2$

3. $\{-2\}$, $x+7=3-x$

4. $\{4\}$, $-3=5-\frac{n}{2}$

5. $\{17\}$, $5x-10=5(x-2)$

6. $\{-100\}$, $5x-10=5(x-2)$

7. $\{0\}$, $x^2-4x=3x+1$

8. $\{0\}$, $(1-x)^2=x+1$

9. $\{-1,9\}$, $|x-4|=5$

10. $\{-3\}$, $|3x|=-9$

Solve equations and check solutions.

11. $7x=56$

12. $12x=72$

13. $x+21=10$

14. $x+12=-7$

15. $4.5+x=-3.1$

16. $-15x=-20$

17. $400=-x$

18. $-2x=-300$

19. $-7x=49$

20. $\frac{y}{8}=11$

21. $a-\frac{1}{6}=-\frac{2}{3}$

22. $-\frac{x}{6}=\frac{2}{9}$

23. $x-\frac{2}{3}=-\frac{5}{6}$

24. $-\frac{2}{3}+y=-\frac{3}{4}$

25. $\frac{3}{4}x=18$

26. $-8.2x=20.5$

27. $-6=y+25$

28. $-6=x+9$

29. $x-4=-19$

30. $t-7.4=-12.9$

31. $-x=28$

32. $-t=-8$

33. $12=-7+y$

34. $x+\frac{1}{3}=\frac{8}{3}$

35. $-\frac{3}{5}y=-\frac{3}{5}$

36. $-\frac{2}{5}x=-\frac{4}{15}$

37. $m-2.8=6.3$

38. $y-5.3=8.7$

39. $\frac{4}{5}x=16$

40. $\frac{3}{4}x = 27$

41. $2x + 3 = 13$

42. $3x - 1 = 26$

43. $2t + 9 = 43$

44. $-5y + 7 = -18$

45. $84 = 7x - 7$

46. $50 = 9t + 1$

Solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations.

47. The payout for playing a single number on a roulette table is 35 times greater than the bet. Cindy receives the payout of $3850. How much was Cindy's bet?

48. A croissant has 53 more calories than a donut. Find the amount of calories in one donut if each croissant has 445 calories.

49. A history textbook costs $90, which is five times cheaper than an anatomy textbook. Find the price of the anatomy text.

50. As of July 28, 2017 Russia contributed 39 crew members for the International Space Station, which is 12 fewer than the number of US astronauts. Find the number of ISS crew members from the United States.

51. According to NASA, the global land-ocean temperature index (average surface temperature of Earth) in 2016 is 1.26 °C greater than it was in 1920. Find the average temperature in 1920 if it was 0.99 °C in 2016.

52. When Kyle went to Europe, his ticket to Berlin was 1.25 times cheaper than the return ticket to New York. How much was the return ticket if the ticket to Berlin cost $480?

53. Damian's age is one year less than twice the Charlotte's age. How old is Charlotte, if Damian is 17 years old?

54. Mary spends 14 minutes on her morning commute, which is 5 minutes shorter than three times the duration of her evening commute. Find how much time she spends on her evening commute.

55. Berat says to Aylin: if you give me 3 apples and then take half of my apples away, then I will be left with 13 apples. How many apples do I have now?

56. Simone says to Adrian: if I give you 5 of my apples and then also half of the remaining amount, then I will have 10 apples left. How many apples do I have now?

Homework 2.1 Answers.

1. No

3. Yes

5. Yes

7. No

9. Yes

11. $\{8\}$

13. $\{-11\}$

15. $\{-7.6\}$

17. $\{-400\}$

19. $\{-7\}$

21. $\left\{-\frac{1}{2}\right\}$

23. $\left\{-\frac{1}{6}\right\}$

25. $\{24\}$

27. $\{-31\}$

29. $\{-15\}$

31. $\{-28\}$

33. $\{19\}$

35. $\{1\}$

37. $\{9.1\}$

39. $\{20\}$

41. $\{5\}$

43. $\{17\}$

45. $\{13\}$

47.
b and p are the bet and the payout amounts respectively, in dollars,
equation: $p = 35b$
solution: $b = 110$

49.
h and a are the prices of the history text and the anatomy text respectively, in dollars,
equation: $h = \frac{a}{5}$
solution: $a = 450$

51.
x and y are the 1920 and the 2016 temperatures respectively, in °C,
equation: $y = x + 1.26$
solution: $x = -0.27$

53.
D and C are the ages of Damian and Charlotte respectively, in years,
equation: $D = (2C) - 1$
solution: $C = 9$

55.
b and a are the amounts of apples Berat has before and after the exchange respectively,
equation: $(b + 3)/2 = a$
solution: $b = 23$

2. Solving Linear Equations

2.1. Linear and Non-linear Equations.

DEFINITION 2.1.1. A *linear equation* is an equation in which each term is either a numerical constant or the product of a numerical constant and the first power of a single variable.

BASIC EXAMPLE 2.1.1. Here are some linear and some non-linear equations.

A linear equation in one variable x:

$$3 + x = 4x - 5$$

A linear equation in two variables x and y:

$$y = 4x - 17$$

A linear equation in three variables x, y, and z:

$$\frac{1}{2}x - 5y + z = 14$$

The following equations are non-linear because they have non-linear terms, highlighted below. This one has a second power of a variable, when only first power is allowed:

$$1 - \mathbf{x^2} = 4x$$

This one has a term with multiple variable factors, when at most one is allowed:

$$5z = \mathbf{3xy} + 7x$$

This one has an absolute value term, which is neither a numerical constant nor a variable product:

$$\mathbf{|x-4|} = 2x + 7$$

This equation is non-linear, but a few algebraic operations can show it is equivalent to a linear equation:

$$\begin{aligned} 2\mathbf{(4-x)} &= 3\mathbf{(x+5)} \\ 8 - 2x &= 3x + 15 && \text{distributed} \end{aligned}$$

This equation is also non-linear, but amazingly it is also equivalent to a linear equation:

$$\begin{aligned} \mathbf{5x^2} + 1 &= 2x\mathbf{(1+2x)} + \mathbf{x^2} \\ \mathbf{5x^2} + 1 &= 2x + \mathbf{4x^2} + \mathbf{x^2} && \text{distributed} \\ \mathbf{5x^2} + 1 &= 2x + \mathbf{5x^2} && \text{combined like terms} \\ \mathbf{5x^2} + 1 - \mathbf{5x^2} &= 2x + \mathbf{5x^2} - \mathbf{5x^2} && \text{subtracted } 5x^2 \text{ on both sides} \\ 1 &= 2x && \text{combined like terms} \end{aligned}$$

2.2. Isolating The Variable. Here we present a procedure for solving linear equations as well as many types of non-linear equations, as long as they turn out to be equivalent to linear equations. It is called *isolating the variable* because it boils down to finding an equivalent (Definition 1.1.2) equation of the form

$$x = c$$

where x is the unknown variable and c is a number. Here x is *isolated* on the left side in the sense that it is there all alone (no other terms, no coefficient), and there is no x on the other side. Even though it is traditional to isolate x on the left side, there is nothing wrong with isolating it on the right: $c = x$.

THEOREM 2.2.1 (Isolating The Variable). The following procedure will isolate the variable x in linear and some non-linear equations.

(1) If the equation contains fractions, multiply both sides by the LCD. Cancel common factors in each fraction to get rid of denominators.
(2) Use the distributive property to get rid of parentheses, and then combine the like terms on both sides.
(3) Use the addition property to cancel the variable terms on one side and the constant terms on the other side of the equation, and combine the like terms again.

By now the equation should have assumed one of the three forms:

- If the equation looks like $ax = b$ then divide both sides of the equation by the coefficient a. Now it looks like $x = c$ and c is the unique solution. The set of solutions can be written as $\{c\}$.
- If the equation looks like $0 = 4$ or something equally wrong, then it is false for all x. There are no solutions: the set of solutions is empty, or $\varnothing$.
- If the equation looks like $0 = 0$, then it is true for all x. Every real number is a solution: the set of solutions is the set of all reals, or $\mathbb{R}$.

When a unique solution exists, it can be checked by substituting it for the variable in the original equation and making sure the equation holds. When the solution set is $\mathbb{R}$, it can be partially checked by substituting a few random numbers for the variable and making sure the equation holds for all of them.

EXAMPLE 2.2.1. Solve the equation

$$3(x-3)+1 = 4+x$$

SOLUTION: No fractions here, so we start by removing parentheses and combining the like terms:

$$\begin{aligned} 3x-9+1 &= 4+x && \text{distributivity} \\ 3x-8 &= 4+x \end{aligned}$$

Then use the addition property of the equation to cancel the terms with x on one side, and the constants on the other side, and combine the like terms again:

$$\begin{aligned} 3x-8-x+8 &= 4+x-x+8 \\ 2x &= 12 \end{aligned}$$

Finally, we divide both sides by the coefficient of the variable term and obtain a solution:

$$\begin{aligned} \frac{2x}{2} &= \frac{12}{2} \\ x &= 6 \end{aligned}$$

To check the answer, we substitute 6 for x in the original equation and simplify:

$$\begin{aligned} 3(x-3)+1 &= 4+x \\ 3(6-3)+1 &= 4+6 \\ 3(3)+1 &= 10 \\ 10 &= 10 \end{aligned}$$

The equation holds, so 6 is a solution.

ANSWER: $\{6\}$

EXAMPLE 2.2.2. Solve the equation

$$4x-(3+x)=-3(1-x)$$

SOLUTION: No fractions here, so we remove the parentheses and combine the like terms:

$$\begin{aligned} 4x-3-x &= -3+3x && \text{distributivity} \\ 3x-3 &= -3+3x \end{aligned}$$

Now we cancel variable terms on one side and constants on the other side, and combine the like terms again:

$$\begin{aligned} 3x-3+3-3x &= -3+3x+3-3x \\ 0 &= 0 \end{aligned}$$

This equation is true for all values of x. In other words, any real number makes the equation true when substituted for x.

ANSWER: $\mathbb{R}$

EXAMPLE 2.2.3. Solve the equation

$$2(7+5x)=1-(3-10x)$$

SOLUTION: No fractions here, so we start by removing parentheses and combining the like terms:

$$\begin{aligned}14+10x &= 1-3+10x\\ 14+10x &= -2+10x\end{aligned}$$

After that we use the addition property to eliminate variable terms on one side and constants on the other side, and combine the like terms again:

$$\begin{aligned}14+10x-10x-14 &= -2+10x-10x-14\\ 0 &= -16\end{aligned}$$

This equation is false for all values of x. In other words, no real number can make the original equation true. There are no solutions.

ANSWER: $\varnothing$

EXAMPLE 2.2.4. Solve the equation

$$\frac{1}{3}x+4=\frac{2}{3}-\frac{1}{2}x$$

SOLUTION: Here the LCD is 6, so we start by multiplying both sides by the LCD:

$$6\left(\frac{1}{3}x+4\right)=6\left(\frac{2}{3}-\frac{1}{2}x\right)$$

As we distribute, the fractions cancel:

$$\begin{aligned}6\cdot\frac{1}{3}x+6\cdot4 &= 6\cdot\frac{2}{3}-6\cdot\frac{1}{2}x\\ 2x+24 &= 4-3x\end{aligned}$$

Now we eliminate variable terms on one side and constants on the other side, combine the like terms, and use the multiplication property to isolate the x variable:

$$\begin{aligned}2x+24+3x-24 &= 4-3x+3x-24\\ 5x &= -20\\ \frac{5x}{5} &= \frac{-20}{5} && \text{divided both sides by 5}\\ x &= -4\end{aligned}$$

To check the answer, we substitute -4 for x in the original equation and simplify:

$$\begin{aligned}
\frac{1}{3}x+4 &= \frac{2}{3}-\frac{1}{2}x \\
\frac{1}{3}\cdot(-4)+4 &= \frac{2}{3}-\frac{1}{2}\cdot(-4) \\
\frac{-4}{3}+4 &= \frac{2}{3}+2 \\
\frac{-4}{3}+\frac{12}{3} &= \frac{2}{3}+\frac{6}{3} && \text{find LCD to add fractions} \\
\frac{8}{3} &= \frac{8}{3}
\end{aligned}$$

The equation holds, so -4 is a solution.

ANSWER: $\{-4\}$

EXAMPLE 2.2.5. Solve the equation

$$\frac{41}{9}=\frac{5}{2}\left(x+\frac{2}{3}\right)-\frac{1}{3}x$$

SOLUTION: Here we have some fractions locked up inside the parentheses, so let us distribute before we try to figure out the LCD:

$$\begin{aligned}
\frac{41}{9} &= \frac{5}{2}\left(x+\frac{2}{3}\right)-\frac{1}{3}x \\
\frac{41}{9} &= \frac{5}{2}x+\frac{5}{2}\cdot\frac{2}{3}-\frac{1}{3}x \\
\frac{41}{9} &= \frac{5}{2}x+\frac{10}{6}-\frac{1}{3}x
\end{aligned}$$

The LCD is 18, so we multiply both sides of the equation by 18 and simplify until all the denominators cancel:

$$\begin{aligned}
18\left(\frac{41}{9}\right) &= 18\left(\frac{5}{2}x+\frac{10}{6}-\frac{1}{3}x\right) && \text{multiply both sides by LCD} \\
\frac{18\cdot 41}{9} &= \frac{18\cdot 5}{2}x+\frac{18\cdot 10}{6}-\frac{18\cdot 1}{3}x && \text{distributed on the right side} \\
82 &= 45x+30-6x && \text{simplified fractions} \\
82 &= 39x+30 && \text{combined like terms}
\end{aligned}$$

Now we finish solving by isolating the variable:

$$\begin{aligned}
82 &= 39x+30 \\
82-30 &= 39x+30-30 \\
52 &= 39x \\
\frac{52}{39} &= \frac{39x}{39} \\
\frac{4}{3} &= x
\end{aligned}$$

We can check the solution by substituting it into the original equation and making sure it holds:

$$\begin{aligned}
\frac{41}{9} &= \frac{5}{2}\left(x+\frac{2}{3}\right)-\frac{1}{3}x \\
\frac{41}{9} &= \frac{5}{2}\left(\frac{4}{3}+\frac{2}{3}\right)-\frac{1}{3}\cdot\frac{4}{3} \\
\frac{41}{9} &= \frac{5}{2}\left(\frac{6}{3}\right)-\frac{4}{9} \\
\frac{41}{9} &= \frac{30}{6}-\frac{4}{9} \\
\frac{41}{9} &= 5-\frac{4}{9} \\
\frac{41}{9} &= \frac{45}{9}-\frac{4}{9} \\
\frac{41}{9} &= \frac{41}{9}
\end{aligned}$$

ANSWER: $\left\{\frac{4}{3}\right\}$

2.3. Applications. We will solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations. As you construct your own solutions to applications, you may be using different variable names, and your equations may also be somewhat different, depending on how you translate English sentences. The final numerical answers, however, should always come out the same, regardless of translation.

EXAMPLE 2.3.1. Tallulah works two jobs. Her full-time job pays four times more than her part-time job, and together they pay 3800 dollars per month. Find how much each job pays.

SOLUTION: Let p be the amount the part-time job pays and let f be the amount the full-time job pays in dollars per month. Let us translate the given statements:

$$\text{full-time job pays 4 times more than part-time job}$$
$$f = 4p$$

And the statement ”together both jobs pay $3800” can be written as

$$\begin{array}{ccccc} \text{full-time pay} & + & \text{part-time pay} & = & \text{total pay} \\ f & + & p & = & 3800 \end{array}$$

Since $f = 4p$, we can substitute $(4p)$ for f into the second equation and obtain a linear equation in one variable:

$$\begin{aligned} f + p &= 3800 && \\ (4p) + p &= 3800 && \text{substitute } (4p) \text{ for } f \\ 5p &= 3800 && \text{combine like terms} \\ \frac{5p}{5} &= \frac{3800}{5} && \text{multiplication property} \\ p &= 760 && \end{aligned}$$

Finally, we substitute 760 for p in the first equation to find that

$$f = 4p = 4(760) = 3040$$

To check the answer, we can verify that the two paychecks add up to the stated total:

$$760 + 3040 = 3800$$

ANSWER:
p and f are part-time and full-time salaries respectively, in dollars

$$\begin{cases} f = 4p \\ f + p = 3800 \end{cases}$$

solution: $p = 760, \quad f = 3040$

EXAMPLE 2.3.2. Rita bought rice and truffles a grocery store, and she paid \$67.5 more for truffles than she paid for the rice. Find the price of each item if the total bill was \$72.

SOLUTION: Let r and t be the prices in dollars of the rice and the truffles respectively. It may be helpful to rephrase given statements to make them easier to translate:

she paid \$67.5 more for truffles than she paid for the rice

the price of truffles is 67.5 dollars greater than the price of rice

$$t = r + 67.5$$

And the other statement

the total bill was \$72

can be translated as

$$r + t = 72$$

Since $t = r + 67.5$, we can substitute $(r + 67.5)$ for t in the last equation, and then solve for r:

$$\begin{aligned}
r + t &= 72 \\
r + (r + 67.5) &= 72 \\
2r + 67.5 &= 72 && \text{combined like terms} \\
2r + 67.5 - 67.5 &= 72 - 67.5 && \text{addition property} \\
2r &= 4.5 && \text{combined like terms} \\
\frac{2r}{2} &= \frac{4.5}{2} && \text{multiplication property} \\
r &= 2.25
\end{aligned}$$

Finally we use the first equation find that

$$\begin{aligned}
t &= r + 67.5 \\
t &= 2.25 + 67.5 \\
t &= 69.75
\end{aligned}$$

ANSWER:

r and t are the prices of rice and truffles respectively, in dollars

$$\begin{cases} t = r + 67.5 \\ r + t = 72 \end{cases}$$

solution: $r = 2.25, \quad t = 69.75$

EXAMPLE 2.3.3. Alice, Bob, and Charlie stand to inherit 12000 dollars, and the will stipulates that the inheritance is to be split in such a way that Bob gets two times more money than Alice, while Charlie should get 1600 dollars less than Alice. Find the inheritance amount for each person.

SOLUTION: Let a, b, and c be the amounts in dollars inherited by Alice, Bob, and Charlie respectively. Bob gets twice as much as Alice, which we can write as

$$b = 2a$$

Charlie gets 1600 dollars less than Alice, which we can write as

$$c = a - 1600$$

Together, their amounts should add up to the total, so we can write this equation:

Alice's share	+	Bob's share	+	Charlie's share	=	total inheritance
a	+	b	+	c	=	12000
a	+	$(2a)$	+	$(a-1600)$	=	12000

This is a linear equation in one variable, so we can solve it the usual way. There are no fractions here, so we start by removing parentheses and combining the like terms:

$$\begin{aligned}
a + 2a + a - 1600 &= 12000 && \text{remove parentheses}\\
4a - 1600 &= 12000 && \text{combined like terms}\\
4a &= 12000 + 1600 && \text{added 1600 on both sides}\\
4a &= 13600 && \text{combined like terms}\\
\frac{4a}{4} &= \frac{13600}{4} && \text{multiplication property}\\
a &= 3400
\end{aligned}$$

So Alice's share is 3400 dollars, Bob's share is $2a = 2 \cdot 3400 = 6800$ dollars, and Charlie's share is $(a - 1600) = 3400 - 1600 = 1800$ dollars. We can see that the original equation holds, since the individual shares add up to the total:

$$3400 + 6800 + 1800 = 12000$$

ANSWER:

a, b, and c are inheritance amounts in dollars for Alice, Bob, and Charlie respectively,

$$\begin{cases} b = 2a \\ c = a - 1600 \\ a + b + c = 12000 \end{cases}$$

solution: $a = 3400, \quad b = 6800, \quad c = 1800$

EXAMPLE 2.3.4. Leilani invested some money into bonds, and the amount of interest earned after one year was 25 times smaller than the original investment. Find the interest and original investment if the total amount in her account is $546.

SOLUTION: Let P be the investment (also called principal) and let I be the interest, in dollars. The interest is 25 times smaller:

$$I = P \div 25$$

The total amount, which is the principal plus the interest, is $546:

$$P + I = 546$$

Substituting $P \div 25$ for I in the second equation gives us an equation in one variable we can solve:

$$\begin{aligned}
P + I &= 546 \\
P + \frac{P}{25} &= 546 && \text{multiply both sides by LCD} \\
25\left(P + \frac{P}{25}\right) &= 25(546) && \text{to get rid of fractions} \\
25P + \frac{25P}{25} &= 25(546) && \text{distributed} \\
25P + P &= 25(546) && \text{simplified fraction} \\
26P &= 25(546) && \text{combined like terms} \\
\frac{26P}{26} &= \frac{25(546)}{26} && \text{multiplication property} \\
P &= 525
\end{aligned}$$

Finally, substituting 525 for P in the first equation gives us

$$I = P \div 25 = 21$$

ANSWER:
P and I are the principal and the interest respectively, in dollars

$$\begin{cases} I = P \div 25 \\ P + I = 546 \end{cases}$$

solution: $P = 525, \quad I = 21$

Homework 2.2.

Solve each equation:

1. $4x = 6x - 28$

2. $-2 = -2m + 12$

3. $-2z - 10 = 5z + 18$

4. $5 + \frac{n}{4} = 4$

5. $0.2x - 0.1 = 0.6x - 2.1$

6. $-11 = -8 + \frac{x}{2}$

7. $\frac{x}{2} - \frac{x}{3} = \frac{x}{6} + 1$

8. $2 - (-3a - 8) = 1$

9. $-5(4x - 3) + 2 = -20x + 17$

10. $66 = 6(6 + 5x)$

11. $\frac{1.37 - x}{4} - 1.29 = -1$

12. $-16n + 12 = 39 - 7n$

13. $3(y - 1) = y + 6$

14. $-(3 - 5n) = 12$

15. $-4x + 10 = -2(3x + 1)$

16. $2(4x - 4) = -20 - 4x$

17. $-(4x - 3) = \frac{15 - 20x}{5}$

18. $-(n + 8) + n = -8n + 2(4n - 4)$

19. $0.4x + 0.2(10 - x) = 3$

20. $-6(8k + 4) = -8(6k + 3) - 2$

21. $\frac{z + 2}{3} = 2z + 3$

22. $\frac{3}{5}(1 + p) = \frac{21}{20}$

23. $2(y - 3.5) = -2(14 - y)$

24. $2b + \frac{9}{5} = -\frac{11}{5}$

25. $\frac{2x - 17}{5} = \frac{17 - 2x}{10}$

26. $-\frac{5}{8} = \frac{5}{4}\left(r - \frac{3}{2}\right)$

27. $\frac{7}{5}z + \frac{3}{5} = -z$

28. $-2(1 - 7p) = 8(p - 7)$

29. $7(y + 3) - 5 = 11y - 4(1 + y)$

30. $-8n - 19 = -2(8n - 3) + 3n$

31. $\frac{3(z - 5)}{2} = \frac{2z + 10}{3}$

32. $\frac{45}{16} + \frac{3}{2}n = \frac{7}{4}n - \frac{19}{16}$

33. $0.25(6t + 2) - 10 = 3t - 7.1$

34. $-76 = 5(1 + 3b) + 3(3b - 3)$

35. $\frac{0.6x}{-12} + 2 = \frac{6 - 0.15x}{3}$

36. $\frac{11}{4} + \frac{3}{4}r = \frac{163}{32}$

37. $\frac{1}{3}n + \frac{29}{6} = 2\left(\frac{4}{3}n + \frac{2}{3}\right)$

38. $-\frac{1}{2}\left(\frac{2}{3}x - \frac{3}{4}\right) - \frac{7}{2}x = -\frac{83}{24}$

Solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations.

39. An éclair has five times as many calories as a latte, and together they have 264 calories. Find how many calories each item has.

40. Cassie has 74 more stickers than Margo, and together they have 400 stickers. Find how many stickers each person has.

41. The width of a rectangular parking lot is 40 feet shorter than its length. Find the dimensions of the lot if its perimeter is 640 feet.

42. The width of a rectangular computer screen is 16/9 times greater than its height. Find the height of the screen if the width is 1920 pixels.

43. Consider an astronomical triangle with sides a, b, and c, such that the length of the side b is 3 light-years longer the length of a, and the length of c is 6 times shorter than the length of a. Find the length of each side if the perimeter of the triangle is 55 light-years.

44. The interest earned by Lin's savings account in 2023 is 50 times smaller than his original investment. Find how much was invested and how much interest was earned if the total amount in the account is $\$1,785$

45. A monthly pass for light rail is 1.2 times cheaper than the a bundle with 20 day passes. Find the price of the monthly pass and the price of the bundle, if the bundle costs $10 more than the monthly pass.

46. Annie made $270 more in tips in June than she did in May. Find how much Annie made in tips in either month if the amount in June was 1.5 times greater than the amount in May.

47. A bread recipe calls for a mix of bleached flour, whole wheat flour, and rye flour. The volume of whole wheat should be two times greater than the volume of bleached, and the volume of rye should be 1 cup less than the volume of bleached. Find the volume in cups for each type of flour if the recipe is asking for 15 cups of flour in total.

48. Three paper supply company employees: Jim, Pam, and Leslie, made a bet about who can make the most sales in a month. Jim won the bet, with Pam making 6 fewer sales than Jim, and Leslie making 7 fewer sales than Jim. Find how many sales each employee has made, if the total amount of sales for the month was 113.

Homework 2.2 Answers.

1. $\{14\}$

3. $\{-4\}$

5. $\{5\}$

7. $\varnothing$

9. $\mathbb{R}$

11. $\{0.21\}$

13. $\left\{\frac{9}{2}\right\}$

15. $\{-6\}$

17. $\mathbb{R}$

19. $\{5\}$

21. $\left\{-\frac{7}{5}\right\}$

23. $\varnothing$

25. $\{17/2\}$

27. $\left\{-\frac{1}{4}\right\}$

29. $\varnothing$

31. $\{13\}$

33. $\{-1.6\}$

35. $\mathbb{R}$

37. $\left\{\frac{3}{2}\right\}$

39.
L and E are calorie amounts in a latte and an éclair respectively

$$\begin{cases} E = 5L \\ E + L = 264 \end{cases}$$

solution: $L = 44, \quad E = 220$

41.
w and l are width and length respectively, in feet

$$\begin{cases} w = l - 40 \\ 2w + 2l = 640 \end{cases}$$

solution: $w = 140, \quad l = 180$

43.
a, b, and c are the side lengths in light-years,

$$\begin{cases} b = a + 3 \\ c = a \div 6 \\ a + b + c = 55 \end{cases}$$

solution: $a = 24, \quad b = 27, \quad c = 4$

45.
m and b the prices of the monthly pass and the bundle respectively, in dollars

$$\begin{cases} m = b \div 1.2 \\ b = m + 10 \end{cases}$$

solution: $m = 50, \quad b = 60$

47.
b, w, and r are the amounts of bleached, whole wheat, and rye flour respectively, in cups,

$$\begin{cases} w = 2b \\ r = b - 1 \\ b + w + r = 15 \end{cases}$$

solution: $b = 4, \quad w = 8, \quad r = 3$

3. Linear Formulas

3.1. Isolating a Variable.

DEFINITION 3.1.1 (Formula). While the word *formula* has no traditional formal definition, this will be our word of choice for equations in more than one variable.

BASIC EXAMPLE 3.1.1. Some formulas useful in applications:

$$P = 2l + 2w$$

is a formula for the perimeter P of a rectangle with length l and width w, and

$$V = lwh$$

is a formula for the volume V of a box with length l, width w, and height h.

DEFINITION 3.1.2. We say that a formula involving a variable y is *solved for* y if it is written in the form

$$y = \text{ expression}$$

where one side of the equation is just y, and the expression on the other side does not contain y at all.

BASIC EXAMPLE 3.1.2. Here are some formulas which are solved for a variable, and some that are not.

E	$=$	mc^2	solved for E
x	$=$	$10y^3 - ab$	solved for x
y	$=$	$\dfrac{4x^2-3x}{5a}+1$	solved for y
y	$=$	$7(4x - \mathbf{y}z)$	**not** solved for y because y appears on the right side
$\mathbf{y^2}$	$=$	$x^2 + 3$	**not** solved for y because neither side is y by itself
$x + \mathbf{40}$	$=$	$y + 4$	**not** solved for x because neither side is x by itself

EXAMPLE 3.1.1. Solve the formula for x:

$$6y - 7x - 10 = 0$$

SOLUTION: We will isolate the variable x by following the same procedure (Theorem 2.2.1) we used to solve linear equations. We just have to pretend that x is the only variable. There are no parentheses, so we use the addition property to isolate the terms with x on one side:

$$\begin{aligned} 6y-7x-10 &= 0 \\ 6y-7x-10-6y+10 &= 0-6y+10 && \text{addition property} \\ -7x &= -6y+10 && \text{combined like terms} \end{aligned}$$

Finally, we divide by the coefficient of the term with x:

$$\begin{aligned} \frac{-7x}{-7} &= \frac{-6y+10}{-7} \\ x &= \frac{-6y+10}{-7} \end{aligned}$$

ANSWER: $x = \dfrac{-6y+10}{-7}$

Note that the answer to the previous example is an *equation*, not a number. Note also that the answer is not unique. In a solved formula such as

$$x = \frac{-6y+10}{-7}$$

the right side can be stated in a variety of ways, all equivalent, and all correct:

$$x = \frac{-6y+10}{-7} = \frac{10-6y}{-7} = \frac{6y}{7} - \frac{10}{7} = \ldots$$

EXAMPLE 3.1.2. Solve the formula for z:

$$5z - x = 3(4-z)$$

SOLUTION: z is locked up in parentheses on the right side, so use distributivity to get rid of these parentheses first:

$$\begin{aligned} 5z-x &= 3(4-z) \\ 5z-x &= 12-3z \end{aligned}$$

Use the addition property to isolate terms with z on one side, and then combine like terms:

$$\begin{aligned} 5z-x+x+3z &= 12-3z+x+3z && \text{addition property} \\ 8z &= 12+x && \text{combined like terms} \end{aligned}$$

Finally, divide by the coefficient of the term with z:

$$\begin{aligned}\frac{8z}{8} &= \frac{12+x}{8}\\ z &= \frac{12+x}{8}\end{aligned}$$

ANSWER: $z = \dfrac{12+x}{8}$

EXAMPLE 3.1.3. Solve the photon energy formula for h:

$$E = \frac{hc}{\lambda}$$

SOLUTION: There is only one term with h, and it is already isolated on the right side, so we just need to use the multiplication property to get rid of the extra factors. Multiply both sides by the reciprocal of each factor on the right side besides h:

$$\begin{aligned}E &= \frac{hc}{\lambda}\\ (E)\frac{\lambda}{c} &= \left(\frac{hc}{\lambda}\right)\frac{\lambda}{c}\\ \frac{E\lambda}{c} &= \frac{hc\lambda}{\lambda c} && \text{cancel common factors on the right}\\ \frac{E\lambda}{c} &= h\end{aligned}$$

ANSWER: $h = \dfrac{E\lambda}{c}$

EXAMPLE 3.1.4. Solve the formula for X:

$$aX = b(X-c)$$

SOLUTION: The variable X is locked up inside the parentheses on the right side, so we distribute first:

$$aX = bX - bc$$

Now we use the addition property to isolate the terms with X on the same side:

$$\begin{aligned} aX - bX &= bX - bc - bX && \text{addition property} \\ aX - bX &= -bc && \text{combined like terms} \end{aligned}$$

We would like to combine the terms with X, but they are not, strictly speaking, similar, having different variable factors. We can, however, distribute and then multiply by the reciprocal of the resulting factor anyway:

$$\begin{aligned} X(a-b) &= -bc && \text{distributivity} \\ \frac{X(a-b)}{a-b} &= \frac{-bc}{a-b} && \text{assuming } (a-b) \neq 0 \\ X &= \frac{-bc}{a-b} \end{aligned}$$

ANSWER: $X = \dfrac{-bc}{a-b}$

Homework 2.3.

Solve the formula for the given variable assuming that all variable expressions involved are non-zero.

1. $ab = c$ for b
2. $g = \frac{h}{i}$ for h
3. $a + c = b$ for c
4. $x - f = g$ for x
5. $P = n(p - c)$ for n
6. $S = L - 2B$ for L
7. $2m + p = 4m + q$ for m
8. $q = 6(L - p)$ for L
9. $\frac{f}{g}x = b$ for x
10. $p = \frac{3y}{q}$ for y
11. $h = vt - 16t^2$ for v
12. $S = \pi rh + \pi r^2$ for h
13. $3x + 2y = 7$ for y
14. $5a - 7b = 4$ for a
15. $4x - 5y = 8$ for y
16. $C = \frac{5}{9}(F - 32)$ for F
17. $4x - 7b = 4$ for b
18. $4x - 5y = 8$ for x
19. $\frac{1}{a} + b = \frac{c}{a}$ for b
20. $lwh = V$ for w
21. $V = \frac{\pi r^2 h}{3}$ for h
22. $rt = d$ for r
23. $ax + b = c$ for a
24. $R = aT + b$ for T
25. $\frac{ym}{b} = \frac{c}{d}$ for y
26. $E = mc^2$ for m
27. $c = \frac{4y}{m + n}$ for y
28. $\frac{rs}{a - 3} = k$ for r
29. $4x - y = 5(x + y) - 1$ for x
30. $10(a + b) = 6(a - 2b)$ for a

Homework 2.3 Answers.

1. $b = \dfrac{c}{a}$

3. $c = b - a$

5. $n = \dfrac{P}{p - c}$

7. $m = \dfrac{p - q}{2}$

9. $x = \dfrac{bg}{f}$

11. $v = \dfrac{h + 16t^2}{t}$

13. $y = \dfrac{7 - 3x}{2}$

15. $y = \dfrac{8 - 4x}{-5}$

17. $b = \dfrac{4 - 4x}{-7}$

19. $b = \dfrac{c}{a} - \dfrac{1}{a}$

21. $h = \dfrac{3V}{\pi r^2}$

23. $a = \dfrac{c - b}{x}$

25. $y = \dfrac{cb}{dm}$

27. $y = \dfrac{c(m + n)}{4}$

29. $x = 1 - 6y$

4. Percent

4.1. Percent Units and Notation.

DEFINITION 4.1.1. 1 *percent*, written also as 1%, is one hundredth of a unit, or 0.01 in decimal representation. The sentence "a part is k% of the whole" or

$$p \text{ is } k\% \text{ of } w$$

can be written as

$$p = \frac{k}{100} \cdot w$$

where p is the part of the whole w, and $k/100$ is the decimal representation of k%.

BASIC EXAMPLE 4.1.1. The decimal representation of k percent, which is $k/100$, may be useful in equations because it is easier to manipulate algebraically. To convert k% into a decimal form, simply divide k by 100. To convert the decimal into percent, multiply it by 100. For example,

k **percent**	**decimal representation**
3%	0.03
35%	0.35
150%	1.5
0.6%	0.006

There are three easy question types which can be translated into the percent equation in terms of the unknown variable x.

Finding the part, knowing the whole and the percentage:

$$\begin{array}{ccccc} \textbf{What} & \text{is} & k\% & \text{of} & w? \\ \mathbf{x} & = & \dfrac{k}{100} & \cdot & w \end{array}$$

Finding the percentage, knowing the whole and the part:

$$\begin{array}{ccccc} p & \text{is} & \textbf{what \%} & \text{of} & w? \\ p & = & \dfrac{\mathbf{x}}{100} & \cdot & w \end{array}$$

Finding the whole, knowing the percentage and the part:

$$\begin{array}{ccccc} p & \text{is} & k\% & \text{of} & \textbf{what number?} \\ p & = & \dfrac{k}{100} & \cdot & \mathbf{x} \end{array}$$

EXAMPLE 4.1.1. What is 14% of 50?

SOLUTION: This translates as

$$x = \frac{14}{100} \cdot 50 = 7$$

Alternatively, we can convert the percent into a decimal form and then solve a similar equation where the fraction is replaced by a decimal:

$$x = 0.14 \cdot 50 = 7$$

ANSWER: 7

EXAMPLE 4.1.2. The number 21 is what percent of 70?

SOLUTION: This question translates as

$$21 = \frac{x}{100} \cdot 70$$

We solve this equation for x:

$$\begin{aligned} (21) \cdot \frac{100}{70} &= \left(\frac{x}{100} \cdot 70\right) \cdot \frac{100}{70} \\ 30 &= x \end{aligned}$$

Note that x is already in percent.

ANSWER: 30%

EXAMPLE 4.1.3. The number 330 is 120% of what number?

SOLUTION: This question translates as

$$\begin{aligned} 330 &= \frac{120}{100} \cdot x && \text{solve this equation for } x \\ (330) \cdot \frac{100}{120} &= \left(\frac{120}{100} \cdot x\right) \cdot \frac{100}{120} \\ 275 &= x \end{aligned}$$

So 330 is 120% of 275.

ANSWER: 275

4.2. Percent Increase and Decrease.

DEFINITION 4.2.1 (Percent Increase and Decrease). We say that a quantity A is obtained by *increasing* the quantity P by $k\%$ if

$$A = P + \frac{k}{100}P$$

Similarly, we say that A is obtained by *decreasing* P by $k\%$ if

$$A = P - \frac{k}{100}P$$

Note that in both cases the percentage is applied to the old quantity in order to obtain the new quantity, not the other way around.

In many cases we need to solve these equations for the old quantity P, and then it is more convenient to use equivalent formulas, obtained by applying the distributive property to the right sides:

$$A = P(1+k/100) \quad \text{percent increase}$$
$$A = P(1-k/100) \quad \text{percent decrease}$$

Percent increase and decrease formulas have many applications. Examples of percent increase include

- the sales tax, when the total price is the original price plus the tax, which is a percentage of the original price
- the price markup, like when the price of a popular item goes up by so many percent
- the bank account or the investment amount increase due to accruing the interest, given the simple interest rate k

Examples of percent decrease include

- the income tax, when the total income is the original amount minus the tax, which is a percentage of the original income
- the price discount, like when a store puts up items for sale at $k\%$ off

EXAMPLE 4.2.1. A department store puts up shirts on sale at 35% discount. Find the original price of a shirt before the discount if the new price is \$26.

SOLUTION: We will need to solve the percent decrease equation for P, which is the original price in dollars. The new price $A = 26$, and the percent decrease is $k = 35$:

$$\begin{aligned} A &= P(1 - k/100) \\ 26 &= P(1 - 35/100) \\ 26 &= P(1 - 0.35) \\ 26 &= 0.65P \\ (26)/0.65 &= (0.65P)/0.65 \\ 40 &= P \end{aligned}$$

ANSWER: \$40

EXAMPLE 4.2.2. A fast food chain raised the price of the beef burger from \$5 to \$6, and lowered the price of the chicken burger from \$6 to \$5. Find the percent increase for the price of the beef burger and the percent decrease for the price of the chicken burger.

SOLUTION: For the beef burger, the old price is $P = 5$ and the new price is $A = 6$ dollars, so the percent increase equation is

$$\begin{aligned} A &= P + \frac{k}{100} \cdot P \\ 6 &= 5 + \frac{k}{100} \cdot 5 \end{aligned}$$

We solve it for k, which is exactly the percent increase:

$$\begin{aligned} 6 - 5 &= 5 + \frac{5k}{100} - 5 && \text{subtracted 5 on both sides} \\ 1 &= \frac{5k}{100} \\ 1 &= \frac{k}{20} && \text{lowest terms} \\ 1 \cdot 20 &= \left(\frac{1}{20}k\right) \cdot 20 \\ 20 &= k \end{aligned}$$

So the price of the beef burger went up by 20%.

The reader may be tempted to think that the price of the chicken burger went down by 20%, but this is not the case. If the old price is $P = 6$ and the new price is $A = 5$ dollars, then the percent decrease equation we have to solve is

$$\begin{aligned}
A &= P - \frac{k}{100} \cdot P \\
5 &= 6 - \frac{k}{100} \cdot 6 \\
5 - 6 &= 6 - \frac{6k}{100} - 6 && \text{subtracted 6 on both sides} \\
-1 &= -\frac{6k}{100} && \text{combined like terms} \\
-1 &= -\frac{3k}{50} && \text{lowest terms} \\
(-1)\left(-\frac{50}{3}\right) &= \left(-\frac{3k}{50}\right)\left(-\frac{50}{3}\right) && \text{multiplied both sides by reciprocal of } -\frac{3}{50} \\
\frac{50}{3} &= k \\
16\tfrac{2}{3} &= k && \text{mixed numbers are great for answers}
\end{aligned}$$

So the price of the chicken burger went down by $16\frac{2}{3}$%.

ANSWER: Up by 20% and down by $16\frac{2}{3}$% respectively

Homework 2.4.

1. What is 6% of 14?

2. What is 9% of 15?

3. 4 is 25% of what number?

4. 15 is 5% of what number?

5. 20 is what percent of 50?

6. 16 is what percent of 64?

7. 100 is what percent of 10?

8. What is 350% of 20?

9. 10.5 is 5% of what number?

10. 1.8 is what percent of 30?

11. What is 0.4% of 5000?

12. 2.2 is 5.5% of what number?

Solve the applications using the percent increase and decrease formulas.

13. Shira earned $208 in tips after 20% income tax. What was her pre-tax income?

14. Timor's pre-tax earnings were $1170. How much money did Timor get after paying 30% income tax?

15. A department store puts dresses up for sale at 30% discount. If the discounted price of a dress is $42, what was the original price before the discount?

16. A used car dealer buys an old van and puts it up for sale with 45% markup. How much did the dealer pay for the van if the price after the markup is $1189?

17. Althea invested $1230 into a savings account with 1.4% monthly interest rate. What is the balance in the account after one month?

18. Uma invested $6500 into a mutual fund with 3.5% monthly interest rate. What is the balance in the account after one month?

19. Paulo paid $5824 for a car, and this price included the 4% sales tax. How much did the car cost before the sales tax was applied?

20. A tea importer's total revenue is $7689.60, and this figure includes the 8% sales tax which needs to be sent to the state. Find the total revenue without the sales tax applied.

21. Omar paid $238 for a webcam during a 15%-off sale. Find the original price of the webcam.

22. Sasha paid $68 for a painting frame during a 20% sale event. Find the original price of the frame.

23. Colin took a $275 pay increase with his promotion. What percent increase does that represent if his old salary was $2250 per month?

24. Aisha took a $700 pay cut when she took a job which allowed her to work from home. What percent decrease does that represent if her new job pays her $2800 per month?

Homework 2.4 Answers.

1. 0.84

3. 16

5. 40%

7. 1000%

9. 210

11. 20

13. $260

15. $60

17. $1247.22

19. $5600

21. $280

23. $12\frac{2}{9}\%$

5. Applications

5.1. Distance and Work. This equation of motion at a constant speed is well known:

$$d = rt$$

Here d is the distance traveled, r is the constant speed, and t is the duration of the motion. For example, a car moving at $r = 45$ miles per hour for $t = 2$ hours will cover the distance of $d = rt = 45 \cdot 2 = 90$ miles. Note that one has to be careful to make a choice of units and stick to it throughout the equation. Here we went with miles for the distance, hours for the duration, and miles per hour for the speed. If the duration was given in, say, minutes, we would have to convert it to hours before using that number in the equation.

We can operate on units of measurement, the way physicists do, to make sure that our equation makes sense. Substituting units for the quantities should produce an equation with the same units on both sides. Note that the equations below are not legitimate algebraic equations, and the square brackets indicate that we are looking at the relationship between the units of measurement, rather than numbers:

$$\begin{aligned} [d] &= [rt] \\ \text{miles} &= \frac{\text{miles}}{\text{hours}} \cdot \text{hours} && \text{hours cancel} \\ \text{miles} &= \text{miles} \end{aligned}$$

This idea generalizes readily for any kind of work done at a constant rate:

$$w = rt$$

In the case with motion, the *work* is simply the distance traveled. But we can also write this type of equation whenever a variable changes at a constant rate over time, even if that variable is not traditionally thought of as *work*. One great way to understand these problems is by thinking about the units of the rate, because the units of its numerator will be the units of work.

$$\begin{aligned} [w] &= [rt] \\ \text{units of work} &= \frac{\text{units of work}}{\text{units of time}} \cdot \text{units of time} && \text{units of time cancel} \\ \text{units of work} &= \text{units of work} \end{aligned}$$

EXAMPLE 5.1.1. Tamika can grade 25 quizzes per hour. Describe the variables in the equation $w = rt$ by stating their units of measurement, and find how many quizzes Tamika can grade in 4 hours.

SOLUTION: The rate of work is measured in quizzes per hour, so we let w be the number of quizzes graded, let r be the rate of grading in quizzes per hour, and let t be the duration of grading in hours. Given $t = 4$ hours, the number of quizzes Tamika can grade is

$$w = 25 \cdot 4 = 100$$

ANSWER:
w is the number of quizzes,
r is in quizzes per hour, t is in hours,
$w = 100$

EXAMPLE 5.1.2. Taylor makes 18 dollars per hour at her job. Describe the variables in the equation $w = rt$ by stating their units of measurement, and find how many dollars Taylor earns in 3.5 hours.

SOLUTION: The earning rate is measured in dollars per hour, so we let w be the dollars earned, let $r = 18$ be the rate in dollars per hour, and let t be the duration of work in hours. Note that in this application we do not refer to Taylor's actual job as "work". Given 3.5 hours, Taylor will earn this many dollars:

$$w = 18 \cdot 3.5 = 63$$

ANSWER:
w is the amount earned in dollars,
r is in dollars per hour, t is in hours,
$w = 63$

EXAMPLE 5.1.3. Uri can walk 7 km in 2 hours. Describe the variables in the equation $w = rt$ by stating their units of measurement, and find how fast Uri walks.

SOLUTION: We need to find Uri's walking speed in km per hour, so we let w be the distance in km, let r be the speed in km per hour, and let t be the duration of the walk in hours. We can plug in what we know and solve the equation for r:

$$\begin{aligned} w &= rt \\ 7 &= r \cdot 2 \\ (7)/2 &= (r \cdot 2)/2 \\ 3.5 &= r \end{aligned}$$

So Uri's walking speed is 3.5 km per hour.

ANSWER:
w is the distance walked in km,
r is in km per hour, t is in hours,
$r = 3.5$

5.2. Applications of $w = rt$. Many problems related to distance and work can be solved by writing an equation for the amount of work completed. For example, when two different vehicles complete the same trip, one moving with speed r_1 for t_1 units of time, the other moving with speed r_2 for t_2 units of time, we can write an equation

$$r_1 t_1 = r_2 t_2$$

to express the fact that the distance covered by each vehicle was the same.

EXAMPLE 5.2.1. Apu takes the express train from Springfield to Capital City, and a regular train on his way back from Capital City to Springfield. The express completes the trip in 4 hours 30 minutes, while the regular train covers the same distance in 6 hours. Find the speed of each train if the express runs 15 miles per hour faster.

SOLUTION: Let x be the speed of the express train and let r be the speed of the regular train in miles per hour. The express is 15 miles per hour faster than the regular train:

$$x = r + 15$$

The distance they cover is the same, so we can also write

$$\text{express distance} \qquad x \cdot 4.5 = r \cdot 6 \qquad \text{regular distance}$$

We have to use 4.5 for the duration of the express trip because we must convert 4 hours 30 minutes into hours for the equation. Now we can substitute $(r+15)$ for x in the second equation and then solve for r:

$$\begin{aligned} x \cdot 4.5 &= r \cdot 6 \\ (r+15) \cdot 4.5 &= r \cdot 6 \\ r \cdot 4.5 + 15 \cdot 4.5 &= r \cdot 6 && \text{distributivity} \\ 4.5r + 67.5 &= 6r && \text{simplifying expressions} \end{aligned}$$

Now we use the addition property to isolate the term with r on the right side:

$$\begin{aligned} 4.5r + 67.5 - 4.5r &= 6r - 4.5r && \text{addition property} \\ 67.5 &= 1.5r && \text{combined like terms} \\ (67.5)/1.5 &= (1.5r)/1.5 && \text{multiplication property} \\ 45 &= r \end{aligned}$$

So the speed of the regular train is 45 miles per hour, and we can find the speed of the express from the first equation:

$$\begin{aligned} x &= r + 15 \\ x &= 45 + 15 \\ x &= 60 \end{aligned}$$

ANSWER:
x and r are the speeds of the express and the regular train respectively, in miles per hour

$$\begin{cases} x = r + 15 \\ 4.5x = 6r \end{cases}$$

solution: $x = 60, \quad r = 45$

Another common situation is when two workers each work towards a common goal, which is to complete a known amount of work. Then we can write that the total work completed is equal to the sum of individual work amounts:

$$\begin{aligned} w_1 + w_2 &= w \\ r_1 t_1 + r_2 t_2 &= w \qquad \text{rewrite } w_1 \text{ and } w_2 \text{ in terms of rates} \end{aligned}$$

EXAMPLE 5.2.2. Kay and July are opening the incoming mail together, with the goal of processing 400 envelopes. Find how much time they need to complete the task if Kay can process 15 envelopes per hour, and July can process 35 envelopes per hour.

SOLUTION: We need to find t, which is the duration of time they both work together, in hours. The total amount of work w is 400 envelopes, so we can write that

$$\begin{array}{ccccc} \text{Kay's work} & + & \text{July's work} & = & 400 \\ 15 \cdot t & + & 35 \cdot t & = & 400 \end{array}$$

This equation can be easily solved for t.

$$\begin{aligned} 50t &= 400 \\ (50t)/50 &= (400)/50 \\ t &= 8 \end{aligned}$$

So it will take 8 hours of them working together to process 400 envelopes.

ANSWER:
t is the duration of work, in hours,
equation: $15t + 35t = 400$
solution: $t = 8$

EXAMPLE 5.2.3. One morning Aristotle starts walking from Athens to Megara, and at the same time Socrates starts on a journey from Megara to Athens. Being younger, Aristotle is walking 1.8 times faster than Socrates. After a 5 hour journey, they meet somewhere in the middle of the way. Find the speed of each traveler if the total distance between Athens and Megara is 42 km.

SOLUTION: Let a and s be the walking speeds of Aristotle and Socrates respectively, in km per hour. What we call *work* here is measured in km, so we are just talking about the distance traveled. Since Aristotle's speed is 1.8 times greater than Socrates' speed, we can write

$$a = 1.8s$$

And since they both walk for 5 hours and then meet in the middle of a 42 km long road, we can also write a distance equation:

$$\begin{array}{ccccc} \text{A's distance} & + & \text{S's distance} & = & \text{total distance} \\ \text{A's speed} \cdot \text{duration} & + & \text{S's speed} \cdot \text{duration} & = & \text{total distance} \\ a \cdot 5 & + & s \cdot 5 & = & 42 \end{array}$$

Now we can substitute $1.8s$ for a in this equation and solve for Socrates' speed s:

$$\begin{array}{rcll} (1.8s) \cdot 5 + (s) \cdot 5 & = & 42 & \\ 9s + 5s & = & 42 & \\ 14s & = & 42 & \text{combined like terms} \\ (14s)/14 & = & (42)/14 & \\ s & = & 3 & \end{array}$$

Now that we know that Socrates was walking at 3 km per hour, we can get Aristotle's speed from the first equation:

$$\begin{array}{rcl} a & = & 1.8s \\ a & = & 1.8(3) \\ a & = & 5.4 \end{array}$$

So Aristotle was walking at 5.4 km per hour.

ANSWER:
a and s are the speeds of Aristotle and Socrates respectively, in km per hour

$$\begin{cases} a = 1.8s \\ 5a + 5s = 42 \end{cases}$$

solution: $a = 5.4, \quad s = 3$

Homework 2.5.

Solve applications of $w = rt$ by describing the variables and their units of measurement.

1. The top speed of a bullet train is 210 km per hour. Find how long it takes to travel 168 km at top speed.

2. The average speed of a cheetah is 16 meters per second. Find how far the cheetah can run in 24 seconds at that speed.

3. A standard computer hard drive rotates at 5400 revolutions per minute. Find the number of revolutions over 10 seconds. (Hint: Convert seconds into minutes.)

4. Morgan's chicken coop produces on average 8 eggs per day. Find how many days are needed to produce 52 eggs.

5. Eve typed up a 4050 word document over two and a half hours. Find Eve's typing rate in words per minute. (Hint: Convert hours into minutes.)

6. Xavier's household used 630 gallons of water over the three winter months. Find the rate of water consumption in gallons per month.

7. Jamie charges \$80 per hour for her paralegal work. Find how long she has to work in order to earn \$2000.

8. Jamar drinks 3.5 cups of coffee per day. Find how long it takes for Jamar to drink 42 cups of coffee.

9. It takes Reza two and a half hours to mow the front lawn. Find how many times he can mow the same lawn in 40 hours.

10. It takes Leela 56 days to inspect 14 buildings. Find the rate of the building inspection in buildings per day.

Solve applications by describing the variables used to represent the quantities of interest, and stating appropriate equations.

11. Olesya paddled for 3 hours upstream and then for 2 hours downstream. When going downstream, her speed was 10 mph greater than her speed when going upstream. The total distance traveled was 30 miles. How fast did Olesya travel upstream and downstream?

12. Fatima rode for 3 hours on the train and then for 2 hours on the bus. The bus is 50 km per hour slower than the train. The total distance traveled by Fatima was 225 km. Find the the speed of the train and the speed of the bus.

13. Ridwan can plant 4 trees per hour, and Sue can plant 5.5 trees per hour. Find how much time they need to plant 285 trees, if they are working together.

14. Quinn runs at 14 km per hour and walks at 6 km per hour. During his weekend exercise, Quinn walks and runs for an equal amount of time. One weekend he ran and walked for a total of 24 km. Find the amount of time he walked.

15. A passenger and a freight train start toward each other at the same time from two points 300 miles apart. If the rate of the passenger train exceeds the rate of the freight train by 15 miles per hour, and they meet after 4 hours, what must the rate of each be?

16. A car and a truck are 276 miles apart and start at the same time to travel toward each other. The car is 5 miles per hour faster than the truck. If they meet after 6 hours, find the rate of each.

17. Lab 1 can process blood samples 3 times faster than Lab 2. Working together, the two labs can process 84 samples over 7 days. Find the rate of work for each Lab, in samples per day.

18. Pump 1 pumps water at the rate which is 10 gallons per minute slower than the rate of Pump 2. With both pumps working together, a 1200 gallon tank can be emptied in 12 minutes. Find the rates in gallons per minute for both pumps.

19. Two automobiles started at the same time from a point, but traveled in opposite directions. Their rates were 25 and 35 miles per hour respectively. After how many hours were they 180 miles apart?

20. Two trains travel toward each other from points which are 195 miles apart. They travel at rate of 25 and 40 miles per hour respectively. If they start at the same time, how soon will they meet?

Homework 2.5 Answers.

1.
w is the distance traveled in km,
r is in km per hour, t is in hours,
$t = 0.8$ (48 minutes)

3.
w is the number of revolutions,
r is in revolutions per minute, t is in minutes,
$w = 900$

5.
w is the number of words,
r is in words per minute, t is in minutes,
$r = 27$

7.
w is wages earned in dollars,
r is in dollars per hour, t is in hours,
$t = 25$

9.
w is lawns mowed,
r is in lawns mowed per hour, t is in hours,
$w = 16$

11.
u and d are the speeds of the upstream and the downstream travel respectively, in mph

$$\begin{cases} d = u + 10 \\ 3u + 2d = 30 \end{cases}$$

solution: $u = 2, \quad d = 12$

13.
t is the duration of work in hours,
equation: $4t + 5.5t = 285$
solution: $t = 30$

15.
p and f are the speeds of the passenger train and the freight train respectively, in mph

$$\begin{cases} p = f + 15 \\ 4p + 4f = 300 \end{cases}$$

solution: $p = 45, \quad f = 30$

17.
r_1 and r_2 are the rates for Lab 1 and Lab 2 respectively, in samples per day

$$\begin{cases} r_1 = 3r_2 \\ 7r_1 + 7r_2 = 84 \end{cases}$$

solution: $r_1 = 9, \quad r_2 = 3$

19.
t is the duration of travel in hours,
equation: $25t + 35t = 180$
solution: $t = 3$

6. Linear Inequalities

6.1. Solution Sets. Recall that we call an equation linear if every term is either a constant or a product of a constant and the first power of a single variable.

DEFINITION 6.1.1. A *linear inequality* is an inequality relation applied to two linear expressions. There are four types of linear inequalities corresponding to the four types of inequality relations: $x < y$, $x > y$, $x \leq y$, and $x \geq y$.

We say that a number is a *solution* for the given inequality in variable x if substituting that number for x makes the inequality true. The collection of all such numbers forms the *solution set* for the inequality.

EXAMPLE 6.1.1. Determine whether 0, 10, and −5, are solutions for the inequality

$$x + 3 < 4 - 2x$$

SOLUTION: Let's try $x = 0$:

$$\begin{aligned} (0) + 3 &< 4 - 2(0) \\ 3 &< 4 \end{aligned}$$

This is true, so 0 is a solution. Let's try $x = 10$ next:

$$\begin{aligned} (10) + 3 &< 4 - 2(10) \\ 13 &< 4 - 20 \\ 13 &< -16 \end{aligned}$$

This is false, so 10 is not a solution. Finally we check $x = -5$:

$$\begin{aligned} (-5) + 3 &< 4 - 2(-5) \\ -2 &< 4 + 10 \\ -2 &< 14 \end{aligned}$$

This is true, so −5 is a solution.

ANSWER: 0 and −5 are solutions, and 10 is not

EXAMPLE 6.1.2. Describe the set of solutions of the inequality $x \geq 6$.

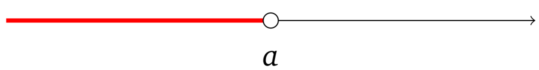
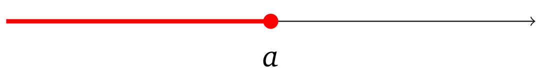
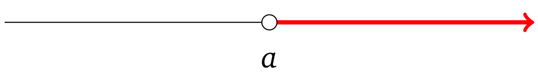
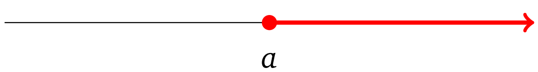
SOLUTION: 6 is clearly a solution, and so is every number greater than 6. At the same time, no number less than 6 is a solution. So the set of solutions is very large: it includes 6, 7, 7.5, and infinitely many other numbers. It would be unhelpful to state it as a roster:

$$\{6, 7, 7.5, 6.12, 100, \ldots\}$$

But the set builder notation provides an unambiguous way of describing this set of numbers:

ANSWER: $\{x \mid x \geq 6\}$

6.2. Interval Notation and Graphs. Inequality solutions in set builder notation are cumbersome, so here we introduce a different notation, which is well-suited for describing large portions of the real line. But before we introduce the notation, let us look at what solutions of linear inequalities look like graphically. In the following table we list the four types of inequalities in one variable x, the corresponding graphs of their solution sets, and the new notation describing these sets.

inequality	**shaded points solve the inequality**	**solution set in interval notation**
$x < a$		$(-\infty, a)$
$x \leq a$		$(-\infty, a]$
$x > a$		(a, ∞)
$x \geq a$		$[a, \infty)$

Note that the graphs of solutions for $x < a$ and $x > a$ leave the point a empty, because $a < a$ and $a > a$ are false, and so a does not belong in either solution set. For a similar reason the graphs of $x \leq a$ and $x \geq a$ fill the point a and mark it as a solution.

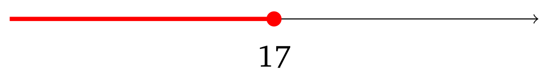
The right column displays the description of the solution set in the interval notation.

DEFINITION 6.2.1 (Interval Notation). The *interval notation* (a, b) describes the set of all real numbers between a and b. When we want to include the endpoints, we use square brackets: $[a, b]$. When we want the interval to extend indefinitely, we use the *infinity* sign: (a, ∞). The nuances are worked out in the following examples.

BASIC EXAMPLE 6.2.1. Let us graph the solution set and describe it in the interval notation:

$$x \leq 17$$

This inequality is true for 17 and all numbers below it:



The interval notation

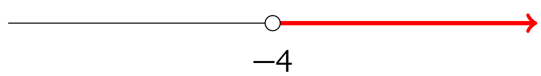
$$(-\infty, 17]$$

uses $-\infty$ for the left endpoint meaning that the set extends indefinitely to the left on the real number line, and 17 for the right endpoint. On the left, the round parenthesis (indicates that $-\infty$ does not solve the inequality, and so not in the solution set. On the right, the square bracket] indicates that 17 does solve the inequality, and so is included in the solution set.

BASIC EXAMPLE 6.2.2. Let's graph the solution set and describe it in the interval notation:

$$x > -4$$

This inequality is true for all numbers above -4, false for -4 and all numbers below it:



The interval notation

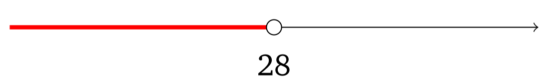
$$(-4, \infty)$$

uses -4 for the left endpoint and ∞ for the right endpoint, meaning that the set extends indefinitely to the right on the real number line. The round parentheses ( ) indicate that neither endpoint solves the inequality.

BASIC EXAMPLE 6.2.3. Let's graph the solution set and describe it in the interval notation:

$$x < 28$$

This inequality is true for all numbers below 28, false for 28 and all numbers above it:



The interval notation

$$(-\infty, 28)$$

uses $-\infty$ for the left endpoint, meaning that the set extends indefinitely to the right on the real number line, and 28 for the right endpoint. The round parentheses ( ) indicate that neither endpoint solves the inequality.

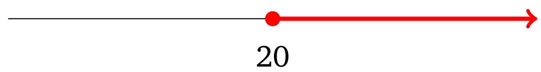
BASIC EXAMPLE 6.2.4. Let's graph the solution set and describe it in the interval notation:

$$20 \leq x$$

Note that we can also write this inequality in an equivalent form

$$x \geq 20$$

Either way, this inequality is true for 20 and all the numbers above 20:



The interval notation

$$[20, \infty)$$

uses 20 for the left endpoint and ∞ for the right endpoint, meaning that the set extends indefinitely to the right on the real number line. The square bracket [on the left indicates that 20 solves the inequality and is an element of the solution set. The round parenthesis) on the right indicates that ∞ does not solve the inequality.

6.3. Addition and Multiplication Properties. Much like we did with the equations, we will solve inequalities by constructing *equivalent* inequalities with well-known solution sets.

DEFINITION 6.3.1. Inequalities are *equivalent* if they have the same solution sets.

THEOREM 6.3.1 (Addition Property for Inequalities). Adding the same number on both sides of an inequality produces an equivalent inequality.

In the following examples, we will describe each solution set by drawing a graph and stating it in the interval notation.

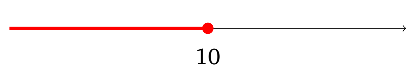
EXAMPLE 6.3.1. Solve the inequality

$$x - 4 \leq 6$$

SOLUTION:

$$\begin{aligned} x - 4 &\leq 6 & & \\ x - 4 + 4 &\leq 6 + 4 & & \text{addition property} \\ x &\leq 10 & & \text{combined like terms} \end{aligned}$$

ANSWER: $(-\infty, 10]$,

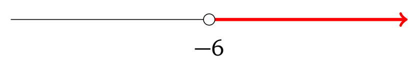


EXAMPLE 6.3.2. Solve the inequality: $4x+3>3(x-1)$

SOLUTION: Just as we did with equations, we will need to isolate the term with the variable on one side, so we start by getting rid of parentheses:

$$\begin{aligned}
4x+3 &> 3(x-1) & \\
4x+3 &> 3x-3 & \text{distributivity} \\
4x+3-3 &> 3x-3-3 & \text{isolating constants on the right side} \\
4x &> 3x-6 & \text{combined like terms} \\
4x-3x &> 3x-6-3x & \text{isolating variable terms on the left side} \\
x &> -6 & \text{combined like terms}
\end{aligned}$$

ANSWER: $(-6,\infty)$,



THEOREM 6.3.2 (Multiplication Property for Inequalities). Multiplying both sides of an inequality by the same **positive** number produces an equivalent inequality. Multiplying both sides of an inequality by the same **negative** number and then **reversing** the inequality sign produces an equivalent inequality.

Formally, the following three inequalities in x are all equivalent to each other for all real numbers a and all positive numbers m:

$$x<a, \quad mx<ma, \quad -mx>-ma$$

Similarly, the following three inequalities in x are all equivalent to each other:

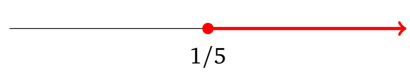
$$x\le a, \quad mx\le ma, \quad -mx\ge -ma$$

EXAMPLE 6.3.3. Solve the inequality: $15x\ge 3$

SOLUTION: Dividing by 15 is the same as multiplying by its reciprocal $1/15$, which is positive, so the inequality sign does not change:

$$\begin{aligned}
15x &\ge 3 \\
(15x)/15 &\ge 3/15 \\
x &\ge 1/5
\end{aligned}$$

ANSWER: $[1/5,\infty)$,

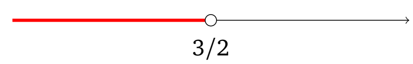


EXAMPLE 6.3.4. Solve the inequality

$$2(1-6x) < -14x+5$$

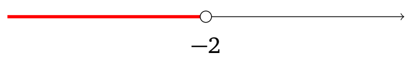
SOLUTION: First we get rid of parentheses and isolate the variable terms on one side, and then we use the multiplication property to isolate the variable:

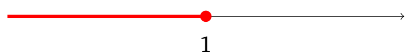
$$\begin{aligned}
2(1-6x) &< -14x+5 \\
2-12x &< -14x+5 \\
2-12x+12x &< -14x+5+12x && \text{added } 12x \text{ to both sides} \\
2 &< -2x+5 && \text{combined like terms} \\
2-5 &< -2x+5-5 && \text{subtracted 5 from both sides} \\
-3 &< -2x && \text{combined like terms} \\
\frac{-3}{-2} &> \frac{-2x}{-2} && \text{dividing both sides by } -2 \text{ reverses the sign} \\
3/2 &> x && \text{this is equivalent to } x < 3/2
\end{aligned}$$

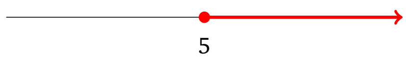
ANSWER: $[-\infty, 3/2)$, 

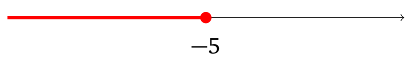
Homework 2.6.

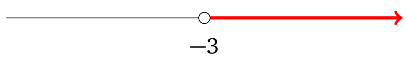
Write an inequality corresponding to the shown solution set, and state it in the interval notation:

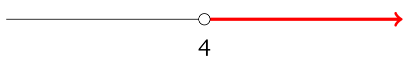
1. 

2. 

3. 

4. 

5. 

6. 

For each inequality, graph the solution set and describe it using the interval notation.

7. $2+x<3$

8. $4y\geq 12$

9. $\dfrac{x}{11}\geq 10$

10. $-2\leq \dfrac{n}{13}$

11. $8+\dfrac{n}{3}\geq 6$

12. $11>8+\dfrac{x}{2}$

13. $2>\dfrac{a-2}{5}$

14. $\dfrac{v-9}{-4}\leq 2$

15. $-47\geq 8-5x$

16. $\dfrac{6+x}{12}\leq -1$

17. $-2(3+k)<-44$

18. $-7n-10\geq 60$

19. $24\geq -6(m-6)$

20. $-8(n-5)\geq 0$

21. $-r-5(r-6)<-18$

22. $-60\geq -4(-6x-3)$

23. $4(2y-3)\leq -44$

24. $2>9-\dfrac{x}{5}$

25. $\dfrac{4}{5}(3x+4)\leq 20$

26. $3(2y-3)>21$

27. $\dfrac{x}{3}+4\leq 1$

28. $\dfrac{2}{3}-\dfrac{x}{5}<\dfrac{4}{15}$

29. $24+4b<4(1+6b)$

30. $-8(2-2n)\geq -16+n$

31. $-5v-5<-5(4v+1)$

32. $-36+6x>-8(x+2)+4x$

33. $3(n+3)+7(8-8n)<5n+5+2$

34. $-(4-5p)+3\geq -2(8-5p)$

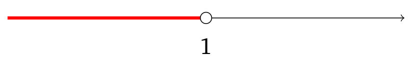
Homework 2.6 Answers.

1. $x < -2$, $(-\infty, -2)$

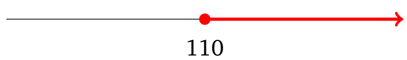
3. $x \geq 5$, $[5, \infty)$

5. $x > -3$, $(-3, \infty)$

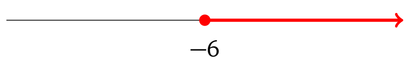
7. $(-\infty, 1)$



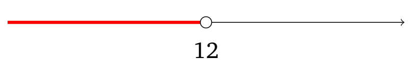
9. $[110, \infty)$



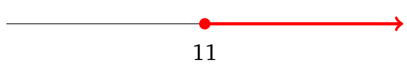
11. $[-6, \infty)$



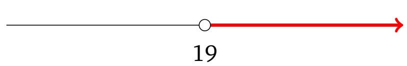
13. $(-\infty, 12)$



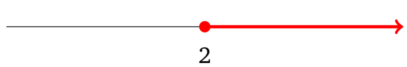
15. $[11, \infty)$



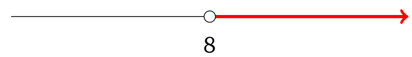
17. $(19, \infty)$



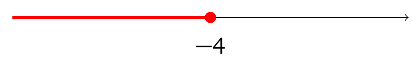
19. $[2, \infty)$



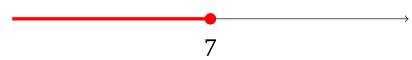
21. $(8, \infty)$



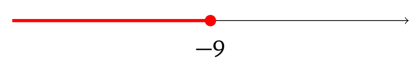
23. $(-\infty, -4]$



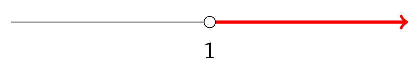
25. $(-\infty, 7]$



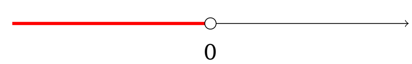
27. $(-\infty, -9]$



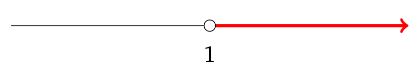
29. $(1, \infty)$



31. $(-\infty, 0)$



33. $(1, \infty)$



Practice Test 2

1. Solve the equation:

$$20x - 7 = -5x - 2$$

2. Solve the equation:

$$7(x+4) - 20 = 4(x+2) + 3x$$

3. Solve the equation:

$$\frac{x}{5} - 1 = \frac{x-1}{5}$$

4. Maria makes 35 dollars per week less than Jeanine, and together they make 1245 dollars per week. How much money does each person make per week?

5. The perimeter of a rectangular cloth is 84 inches, and one side of the rectangle is 2.5 shorter than the long side. Find the dimensions of the cloth.

6. Solve the equation:

$$x - 11 + 2(x-2) = 6(5-x)$$

7. Solve the formula for D:

$$\frac{AD}{3} = \frac{7}{2B}$$

8. Solve the formula for y:

$$-5y + 4x - 12 = 18$$

9. Solve the formula for w:

$$w + 6x = -4(2y - w)$$

10. What is 14% of 280?

11. 7 is what percent of 5?

12. 384 is 96% of what number?

13. The monthly cost of living in a metropolitan area went down from 1400 to 1330. What percent decrease does that represent?

14. A savings account has \$571.20 at the end of the year, after the 2% simple interest is added. How much money was originally invested into the account at the beginning of the year?

15. Train A, traveling 70 miles per hour, leaves Westfall heading toward East River, 260 miles away. At the same time Train B, traveling 60 miles per hour, leaves East River heading toward Westfall. How soon do the two trains meet?

16. A bus leaves the station at noon. Two hours later a car leaves the same station, going 10 miles per hour faster than the bus, and follows the same route. Find the speed of each vehicle if the car catches up to the bus 8 hours after the bus leaves the station.

(Hint: Both vehicles end up covering the same distance, although the car travels over a shorter period of time.)

17. Solve the inequality:

$$8(x+1) \geq 4x - 24$$

18. Solve the inequality:

$$7x - 3 > \frac{1}{2}(15x + 4)$$

Practice Test 2 Answers.

1. $\{1/5\}$

2. $\mathbb{R}$

3. $\varnothing$

4.
m and j are amounts for Maria and Jeanine respectively, in dollars per week

$$\begin{cases} m = j - 35 \\ m + j = 1245 \end{cases}$$

solution: $m = 605, \quad j = 640$

5.
l and w are length and width of the cloth, in inches

$$\begin{cases} 2l + 2w = 84 \\ w = l \div 2.5 \end{cases}$$

solution: $l = 30, \quad w = 12$

6. $\{5\}$

7. $D = \dfrac{21}{2AB}$

8. $y = \dfrac{30 - 4x}{-5}$

9. $w = \dfrac{6x + 8y}{3}$

10. 39.2

11. 140%

12. 400

13. 5% decrease

14. \$560

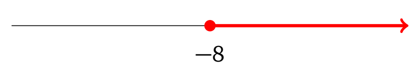
15.
t is the duration of travel until trains meet, in hours,
equation: $70t + 60t = 260$
solution: $t = 2$

16.
b and c are the speeds of the bus and the car respectively, in mph

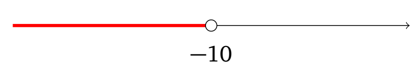
$$\begin{cases} c = b + 10 \\ b \cdot 8 = c \cdot 6 \end{cases}$$

solution: $b = 30, \quad c = 40$

17. $[-8, \infty)$



18. $(-\infty, -10)$



CHAPTER 3

Graphing

1. Reading and Constructing Graphs

1.1. Cartesian Plane.

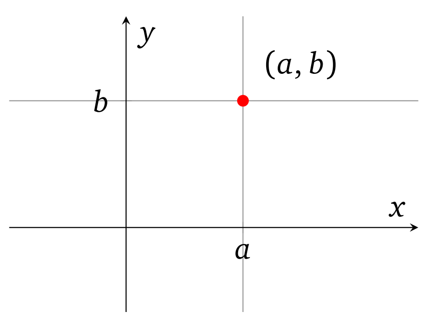
DEFINITION 1.1.1 (Ordered Pair). An *ordered pair* is a pair of numbers with a definite order to them. The notation (a, b) denotes an ordered pair with number a first and number b second. Here a is the *x coordinate*, or the *first coordinate*, or the *first component* of the pair, and b is the *y coordinate*, or the *second coordinate*, or the *second component* of the pair.

Two ordered pairs are *equal* if their first components are equal and their second components are equal.

BASIC EXAMPLE 1.1.1. The pairs $(10, 25)$ and $(5+5, 5^2)$ are equal.

The pairs $A = (10, 25)$ and $B = (10, 10)$ are distinct even though their first components are equal, since the second component of A is 25, and it is not equal to the second component of B, which is 10.

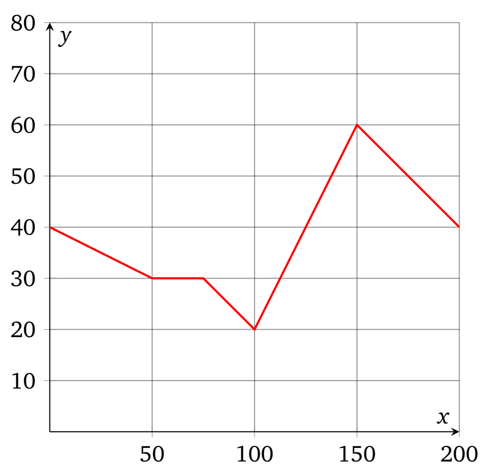
DEFINITION 1.1.2. A *Cartesian plane* is an infinite flat plane equipped with a Cartesian coordinate system. In this system, an ordered pair of real numbers (a, b) corresponds to the point of intersection of two lines: the vertical line through the point a on the x-axis, and the horizontal line through the point b on the y-axis.

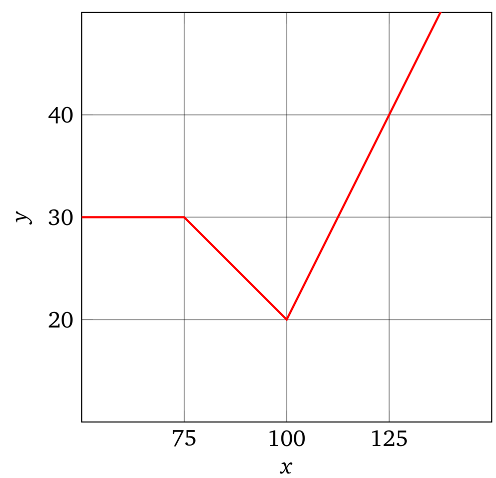


Note that the distance from the point (a, b) to the x-axis is $|b|$, while the distance to the y-axis is $|a|$.

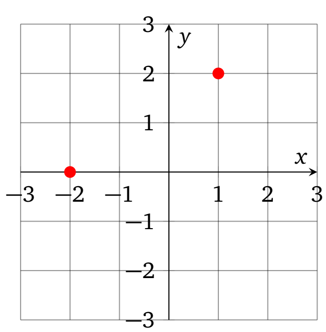
DEFINITION 1.1.3. We can *graph a set of points* on a coordinate plane by shading all points in that set with a specific color.

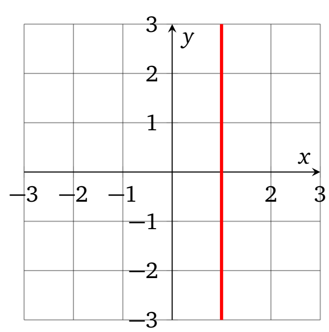
BASIC EXAMPLE 1.1.2. There are two major visual styles for the coordinate plane. The illustration on the left shows the coordinate axes as thick black lines, and is typically used when the origin $(0,0)$ is visible on the graph. The illustration on the right is styled like a box and is often used when showing the axes is inconvenient.

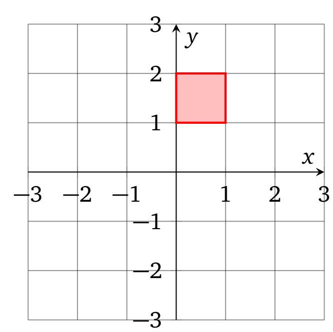




BASIC EXAMPLE 1.1.3. Finite sets of points look like collections of dots. The graph on the left shows the set containing points $(-2,0)$ and $(1,2)$:





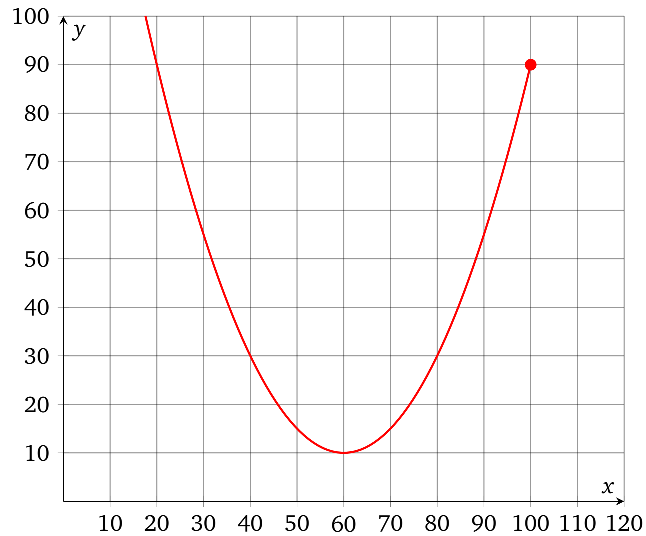


Infinite sets of points often look like lines and curves. The middle graph shows the set containing every point with x coordinate equal to 1.

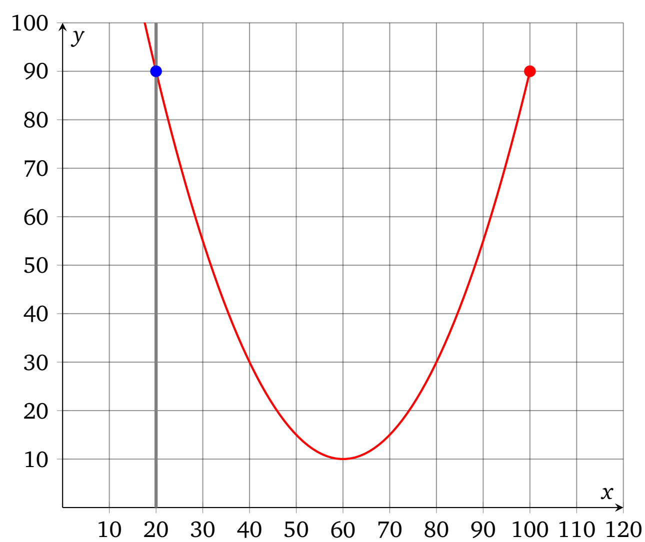
Even larger sets of points, usually defined by inequalities, often appear as solid shaded regions of the plane. The graph on the right shows the set of all points with x coordinate between 0 and 1 and y coordinate between 1 and 2.

EXAMPLE 1.1.1. Consult the given graph and find

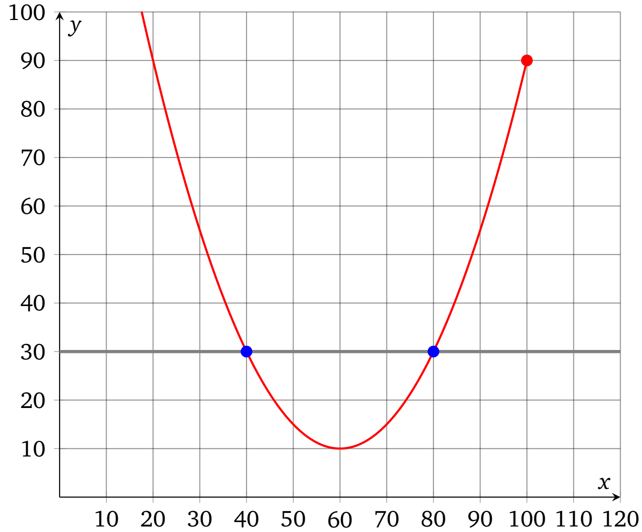
(1) all points with x coordinate 20
(2) all points with y coordinate 30
(3) the point with the lowest y coordinate
(4) the point with the highest x coordinate



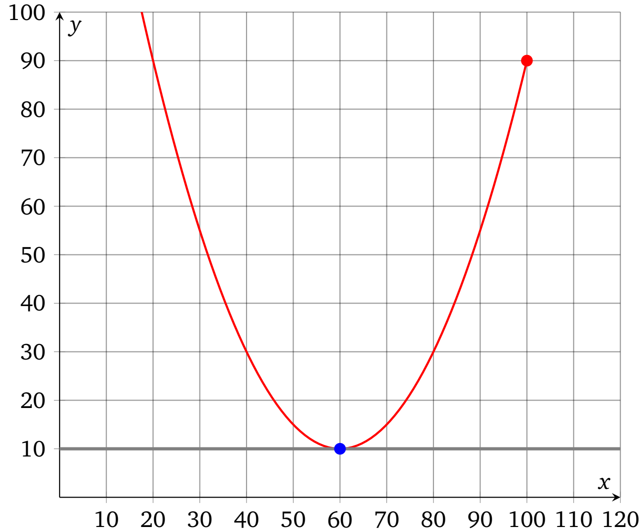
SOLUTION: To find the all points with x coordinate 20, we can follow the vertical line consisting of points with x coordinate equal to 20 and see where this line meets the graph. In our case, the only such point is $(20, 90)$:



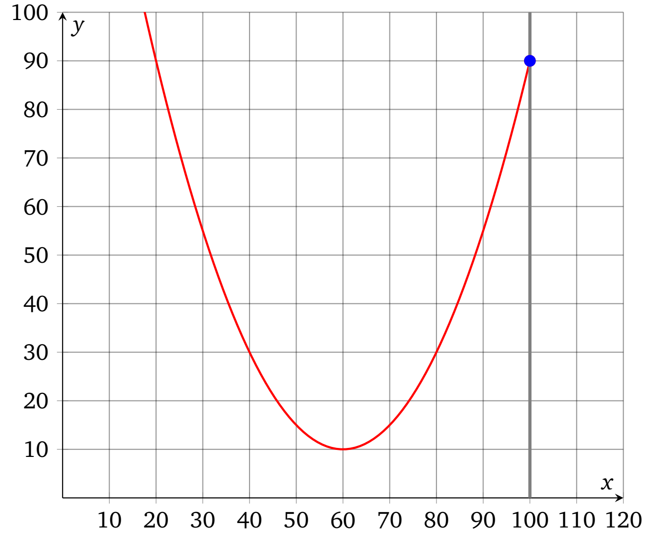
To find the all points with y coordinate 30, we can follow the horizontal line consisting of points with y coordinate equal to 30 and see where this line meets the graph. In our case, we can see two such points, $(40, 30)$ and $(80, 30)$:



To find the point with the lowest y coordinate, we look at how low the graph goes, since points which appear to be down in the bottom have lower y coordinates. In our case, the graph goes as low as $y = 10$, and the only point this far down is $(60, 10)$:



To find the point with the highest x coordinate, we look at how far to the right the graph goes, since points which appear to be on the right side have higher x coordinates. In our case, the graph goes as far to the right as $x = 100$, and the only point this far on the right is $(100, 90)$:



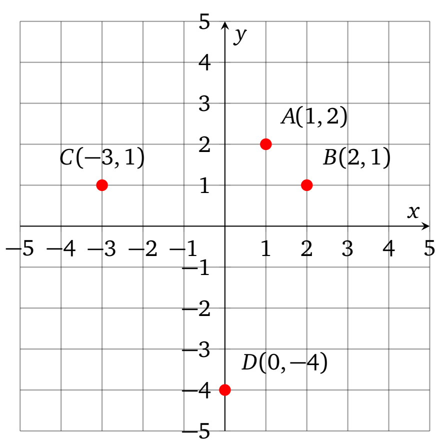
ANSWER:

(1) $(20,90)$
(2) $(40,30)$ and $(80,30)$
(3) $(60,10)$
(4) $(100,90)$

EXAMPLE 1.1.2. Plot points $A(1,2)$, $B(2,1)$, $C(-3,1)$, and $D(0,-4)$ on the same plane.

SOLUTION: Note that A and B are different points: the first coordinate of $A(1,2)$ is 1, so A is located above 1 on the x-axis, whereas $B(2,1)$ is located above 2 on the x-axis.

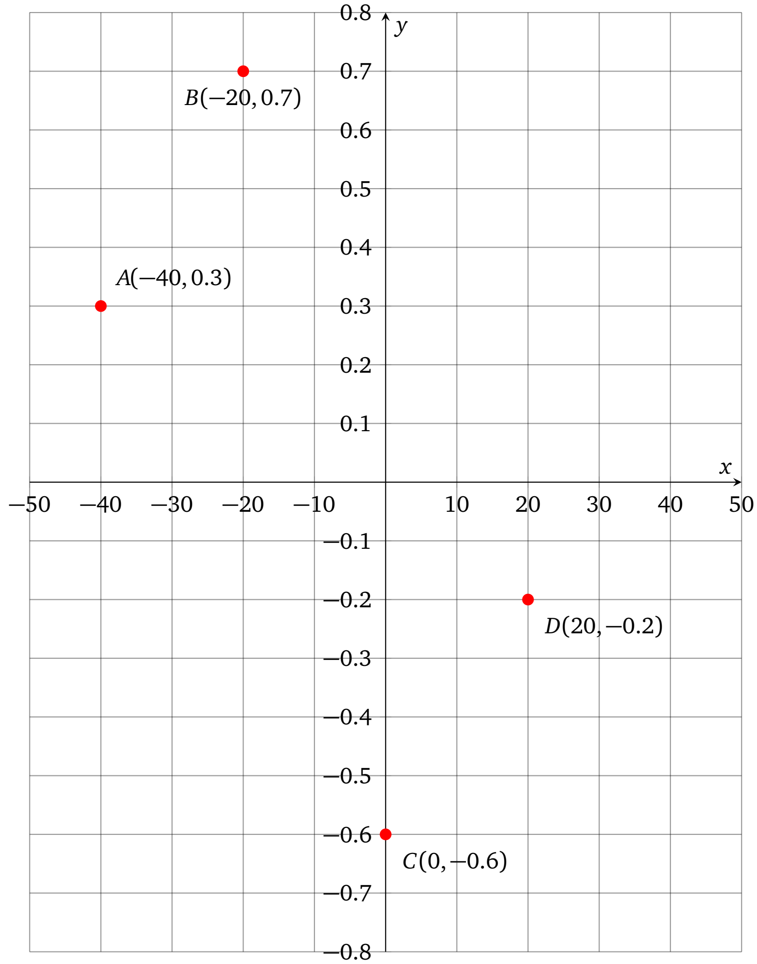
ANSWER:



EXAMPLE 1.1.3. Plot points $A(-40, 0.3)$, $B(-20, 0.7)$, $C(0, -0.6)$, and $D(20, 0.2)$ on the same plane.

SOLUTION: We may have to scale the coordinate axes in order to make the points spread out in a visually appealing fashion. In this case, the x coordinate ranges from −40 to 20, while the absolute value of the y coordinate never goes above 1.0.

ANSWER:



Homework 3.1.

Plot all given points on the same coordinate plane.

1.

$A(1,3)$
$B(-2,4)$
$C(-1,-1)$
$D(2,0)$

2.

$A(0,1)$
$B(2,-3)$
$C(1,-5)$
$D(-5,0)$

3.

$A(-1,-60)$
$B(0,-20)$
$C(1,80)$
$D(2,90)$

4.

$A(-120,0)$
$B(-20,-2)$
$C(40,2)$
$D(100,-3)$

5.

$A(-0.4,0.4)$
$B(0.1,-0.6)$
$C(0.5,0.1)$
$D(0.7,0)$

6.

$A(0,-0.8)$
$B(-0.5,-0.7)$
$C(0.4,0)$
$D(-0.4,0.5)$

7.

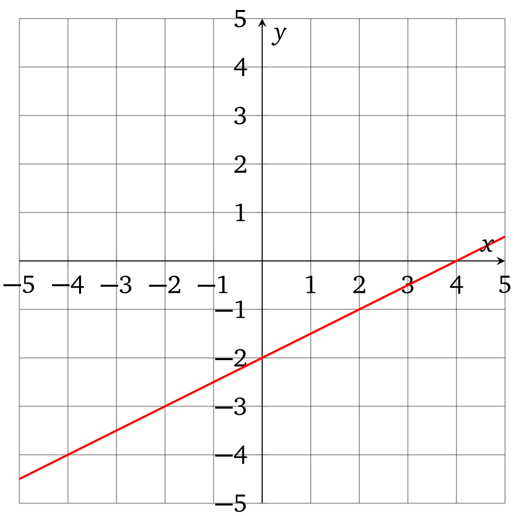
$A(10,15)$
$B(-5,-5)$
$C(15,-10)$
$D(-5,15)$

8.

$A(0,0)$
$B(0,400)$
$C(200,300)$
$D(400,0)$

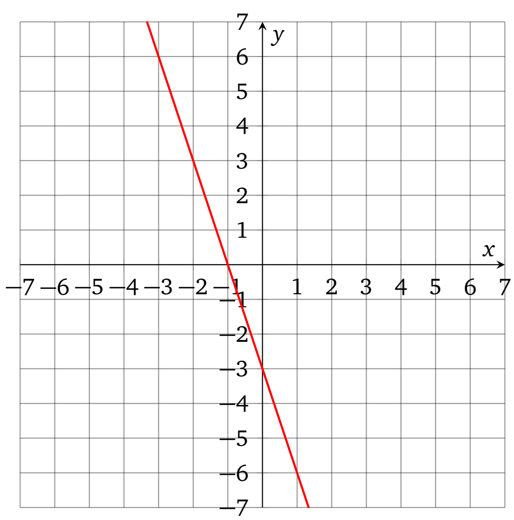
For each graph shown, find coordinates of the points with required properties.

9.



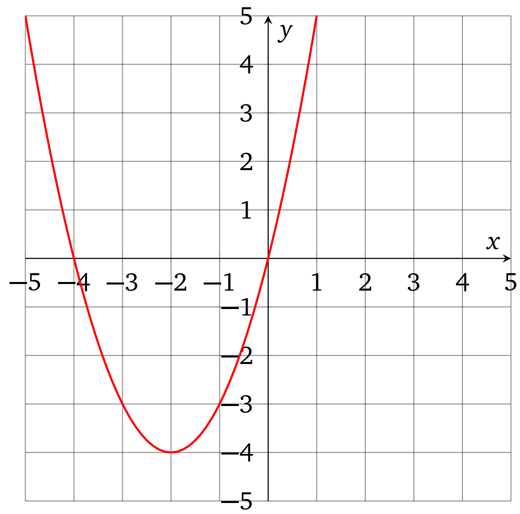
(1) all points with x coordinate zero
(2) all points with y coordinate zero
(3) all points with x coordinate −4
(4) all points with y coordinate −1

10.



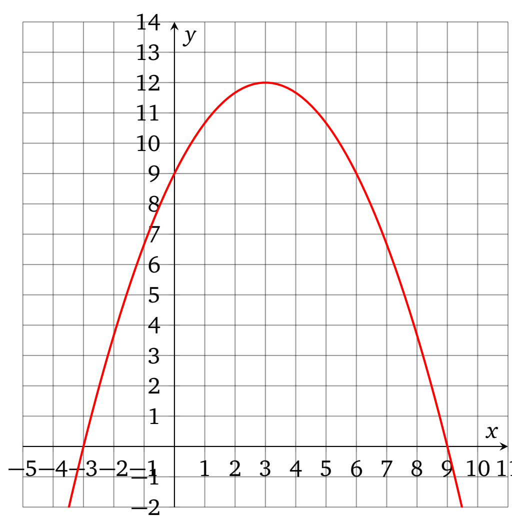
(1) all points with x coordinate zero
(2) all points with y coordinate zero
(3) all points with x coordinate −2
(4) all points with y coordinate −6

11.



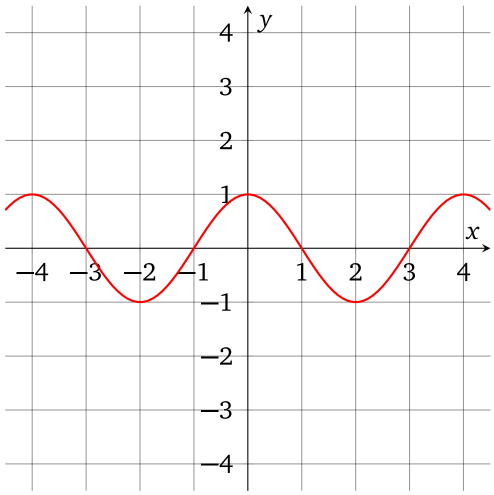
(1) all points with x coordinate zero
(2) all points with y coordinate zero
(3) all points with x coordinate −3
(4) the point with the lowest y coordinate

12.



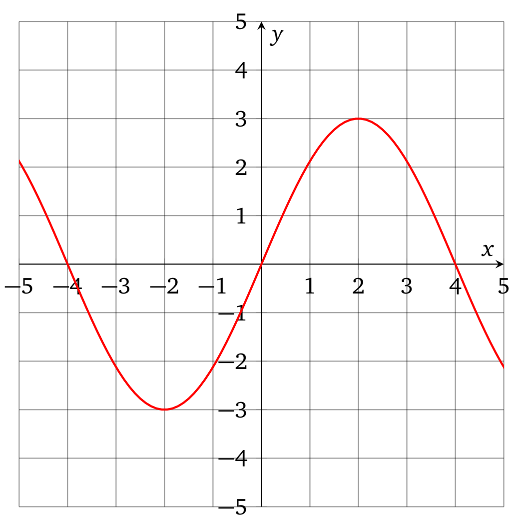
(1) all points with x coordinate zero
(2) all points with y coordinate zero
(3) all points with x coordinate 6
(4) the point with the highest y coordinate

13.



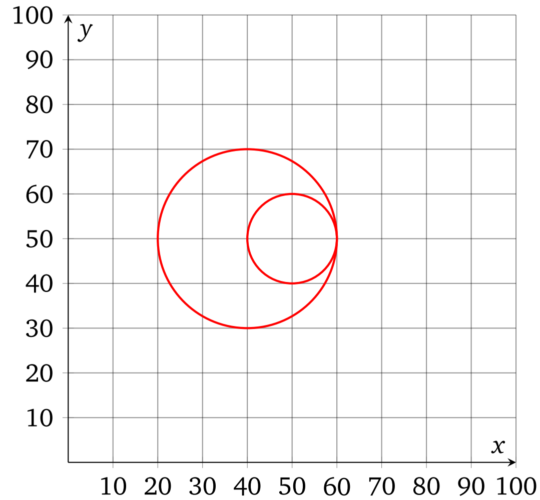
(1) all points with x coordinate zero
(2) all points with y coordinate zero
(3) all points with x coordinate 2
(4) all points with the lowest y coordinate

14.



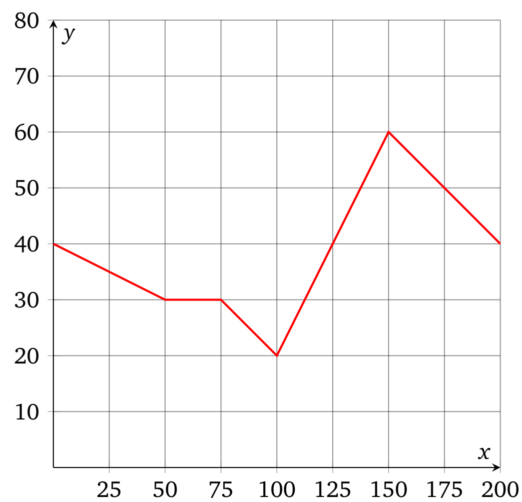
(1) all points with x coordinate zero
(2) all points with y coordinate zero
(3) all points with x coordinate -2
(4) the point with the highest y coordinate

15.



(1) all points with x coordinate 40
(2) all points with y coordinate 50
(3) the point with the highest x coordinate
(4) the point with the lowest y coordinate

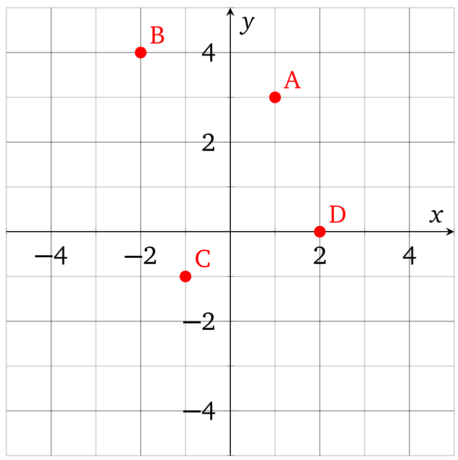
16.



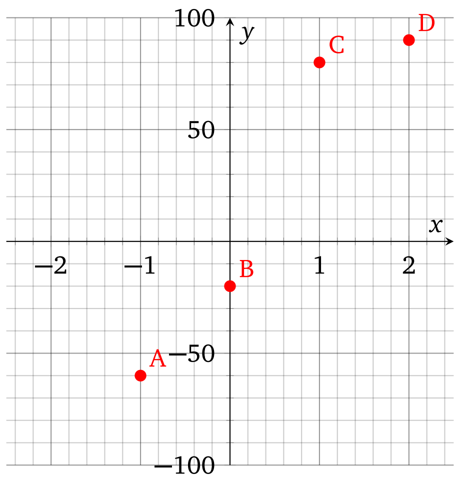
(1) all points with x coordinate 125
(2) all points with y coordinate 40
(3) the point with the lowest y coordinate
(4) the point with the highest y coordinate

Homework 3.1 Answers.

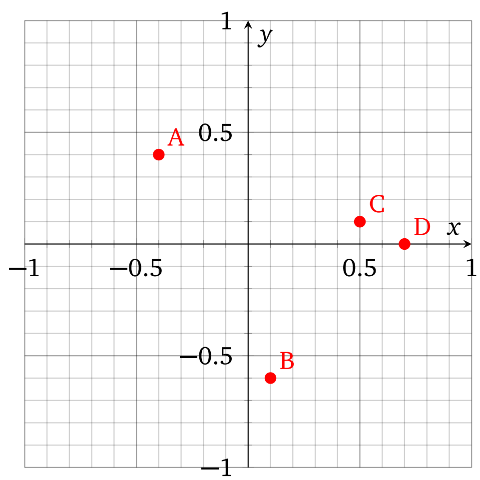
1.



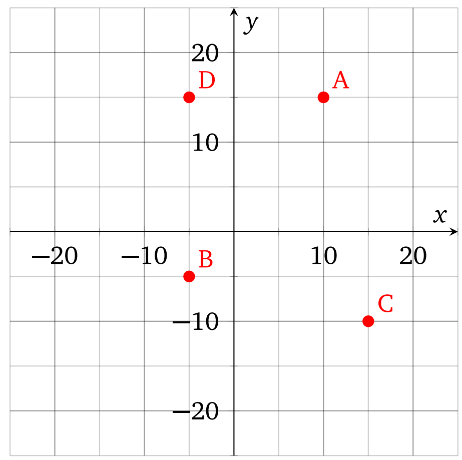
3.



5.



7.



9.

(1) $(0, -2)$
(2) $(4, 0)$
(3) $(-4, -4)$
(4) $(2, -1)$

11.

(1) $(0, 0)$
(2) $(-4, 0)$ and $(0, 0)$
(3) $(-3, -3)$
(4) $(-2, -4)$

13.

(1) $(0, 1)$
(2) $(-3, 0)$, $(-1, 0)$, $(1, 0)$, $(3, 0)$
(3) $(2, -1)$
(4) $(-2, -1)$ and $(2, -1)$

15.

(1) $(40, 30)$, $(40, 50)$, $(40, 70)$
(2) $(20, 50)$, $(40, 50)$, $(60, 50)$
(3) $(60, 50)$
(4) $(40, 30)$

2. Graphing Linear Equations

2.1. Solution Sets. Recall that equation is a particular kind of relation, which we informally defined as a true-or-false statement about numbers. Consider an equation in two variables x and y. It may be true for some ordered pairs (a, b) and false for others.

DEFINITION 2.1.1. A *solution to an equation in two variables* x and y is any ordered pair (a, b) such that substituting a for x and b for y makes the equation true. The set of all such pairs is the *solution set* for that particular equation.

EXAMPLE 2.1.1. Determine whether $(-5, 4)$ is a solution for the equation $2x + 3y = 3$

SOLUTION: The left side of the equation is

$$\begin{aligned} 2x + 3y &= 2(-5) + 3(4) \\ &= -10 + 12 \\ &= 2 \end{aligned}$$

while the right side is 3. The sides are unequal, the equation is false, so this is not a solution.

ANSWER: No

EXAMPLE 2.1.2. Determine whether $(3, -2)$ a solution for the equation $2x^2 + y^3 = 4x + y$

SOLUTION: The left side of the equation is

$$\begin{aligned} 2x^2 + y^3 &= 2(3)^2 + (-2)^3 \\ &= 2(9) + (-8) \\ &= 10 \end{aligned}$$

while the right side is

$$\begin{aligned} 4x + y &= 4(3) + (-2) \\ &= 10 \end{aligned}$$

The sides are equal, the equation is true, so this is a solution.

ANSWER: Yes

BASIC EXAMPLE 2.1.1. Some equations have empty solution sets, for example

$$x + y = x + y + 17$$

Some equations have solution sets which consist of all possible ordered pairs of numbers, for example

$$x + y = x + y$$

But most practically useful equations have solution sets which include some pairs, and exclude others.

EXAMPLE 2.1.3. Describe the solution set for the equation $y = x$.

SOLUTION: Any pair of the form (a, a), where a is a real number, will solve the equation. At the same time, any pair of distinct numbers (a, b) such that $a \neq b$ will make the equation false, so the solution set consists of all pairs (a, a) where the components are equal to each other. For example, $(1, 1)$ is a solution, and so is $(3, 3)$, $(-10, -10)$, and so on. As is typically the case with equations in two variables, the solution set is infinite.

ANSWER: The set of pairs (a, a), for each real number a.

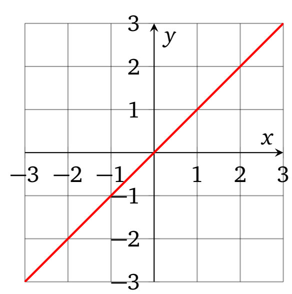
2.2. Graphing Solution Sets.

DEFINITION 2.2.1. A *graph of a relation*, and in particular of an equation, consists of all points on the plain for which that particular relation is true.

EXAMPLE 2.2.1. Graph the equation $x = y$.

SOLUTION: The reader can refer to the example 2.1.3 for the description of the solution set, which consists of all the points (a, a), for each real number a. As we start plotting some of the solutions on the Cartesian plane, we realize they are all located on the so-called main diagonal.

ANSWER:



EXAMPLE 2.2.2. Graph the equation $y = 5 - x^2$.

SOLUTION: Whenever an equation is solved for y, we can attempt to sketch it by evaluating it for convenient values of x, and then connecting the dots, so to speak.

If $x = 0$, then $y = 5 - (0)^2 = 5 - 0 = 5$

If $x = 0.5$ then $y = 5 - (0.5)^2 = 5 - 0.25 = 4.75$

If $x = 1$, then $y = 5 - (1)^2 = 5 - 1 = 4$

If $x = 2$, then $y = 5 - (2)^2 = 5 - 4 = 1$

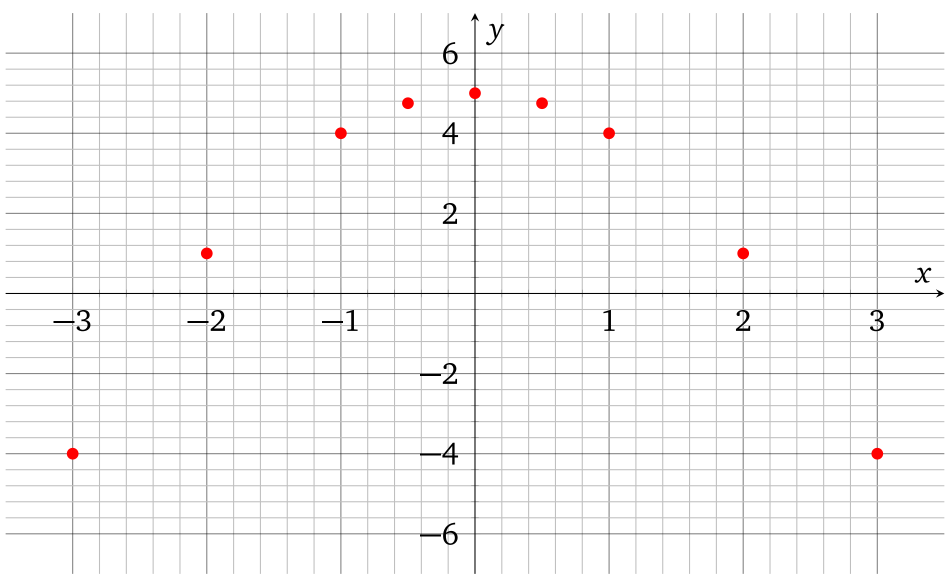
If $x = 3$, then $y = 5 - (3)^2 = 5 - 9 = -4$

If $x = -0.5$, then $y = 5 - (-0.5)^2 = 5 - 0.25 = 4.75$

In fact, negative x will result in the same y values as their positive inverses, so we also get the points $(-1, 4)$, $(-2, 1)$ and $(-3, -4)$. We can summarize the points we have found in a table of the corresponding x and y coordinates:

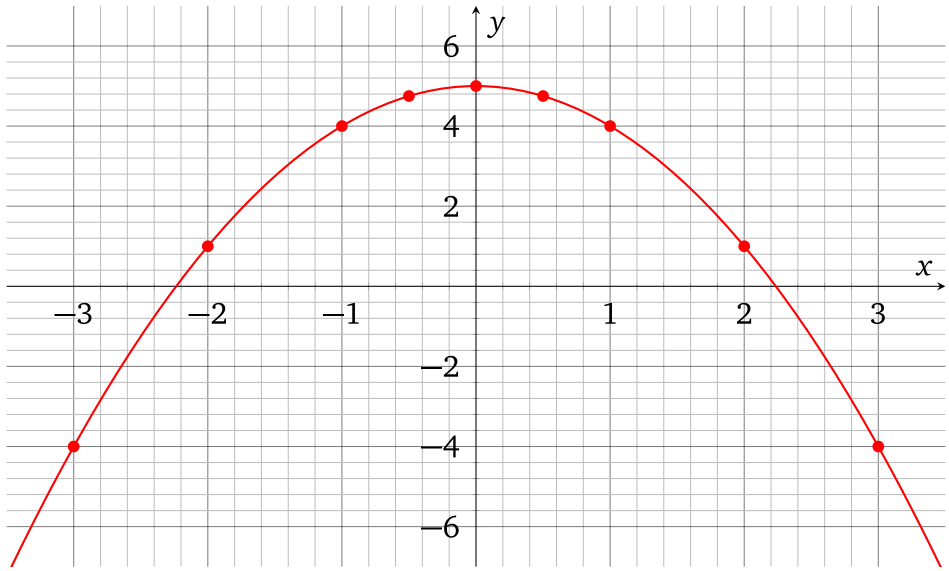
x	-3	-2	-1	-0.5	0	0.5	1	2	3
y	-4	1	4	4.75	5	4.75	4	1	-4

We plot these points on the Cartesian plane:



At last, we draw a smooth curve through the dots.

ANSWER:



2.3. Graphing Lines.

THEOREM 2.3.1. A graph of a linear equation in variables x and y

$$Ax + By = C$$

where not both A and B are zero, is always a straight line. Conversely, every straight line on the Cartesian plane is the solution set of some linear equation in this form.

So the task of graphing a linear equation can be reduced to plotting just two solutions, and then drawing a straight line through these points.

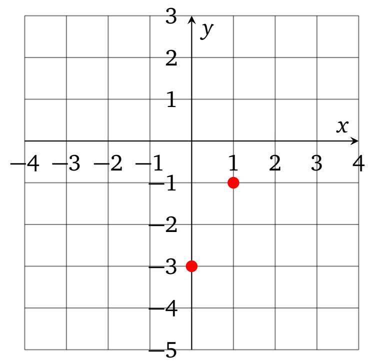
EXAMPLE 2.3.1. Graph the equation $y = 2x - 3$

SOLUTION: We can substitute any two numbers for x, but the best choices would make it easy to find the corresponding y, so we will stick to small integers.

If $x = 0$, then $y = 2(0) - 3 = -3$

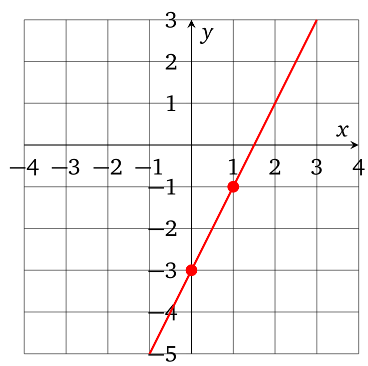
If $x = 1$, then $y = 2(1) - 3 = -1$

We plot the points $(0, -3)$ and $(1, -1)$ on the plane:



Finally, we draw a straight line through the points we found.

ANSWER:



EXAMPLE 2.3.2. Graph the equation $3x - 5y = 10$

SOLUTION: Guessing the solutions for this equation is inconvenient, so we solve it for y. The result is an equivalent equation which makes it really easy to find solution pairs.

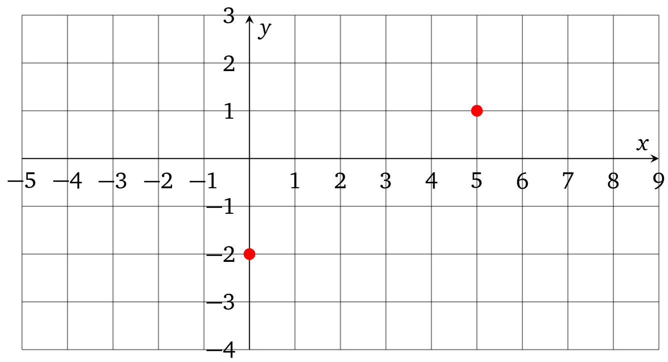
$$\begin{aligned} 3x - 5y &= 10 \\ -5y &= 10 - 3x \\ \frac{-5y}{-5} &= \frac{10-3x}{-5} \\ y &= -2 + \frac{3}{5}x \qquad \text{distributivity} \end{aligned}$$

It looks like we will have easier time if we substitute multiples of 5 for x.

If $x = 0$, then $y = -2 + \frac{3}{5}(0) = -2$

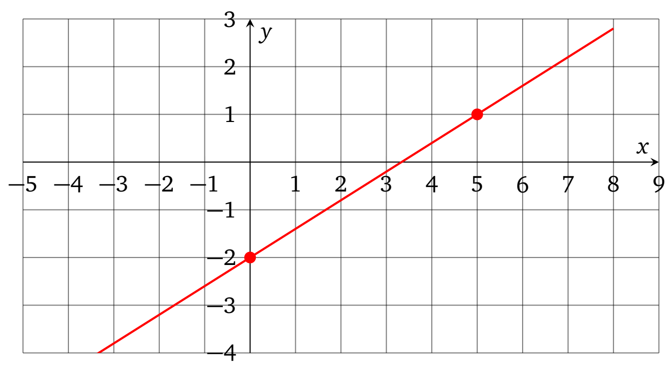
If $x = 5$, then $y = -2 + \frac{3}{5}(5) = -2 + 3 = 1$

We plot the points $(0, -2)$ and $(5, 1)$ on the plane:



Finally, we draw a straight line through the points we found.

ANSWER:



Homework 3.2.

Graph the equation and highlight two distinct points on the line.

1. $y = x + 2$

2. $y = x - 2$

3. $y = 2x$

4. $y = \frac{1}{3}x$

5. $y = -2x + 1$

6. $y = -x - 3$

7. $y = -\frac{4}{5}x - 3$

8. $y = \frac{3}{2}x - 5$

9. $x = -1$

10. $y = 4$

11. $x + y = -1$

12. $x - y = -3$

13. $x + 5y = -15$

14. $4x + y = 5$

15. $8x - y = 5$

16. $3x + 4y = 16$

17. $y + 3 = 0$

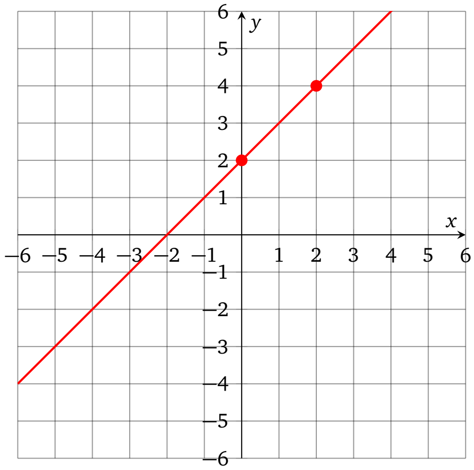
18. $x - 2 = 0$

19. $2x + 3y = 0$

20. $-4x + 3y = 0$

Homework 3.2 Answers.

1.



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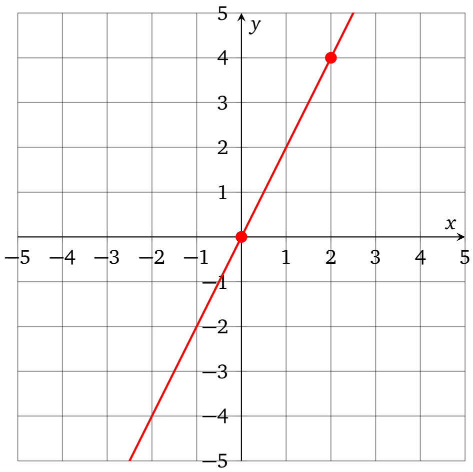

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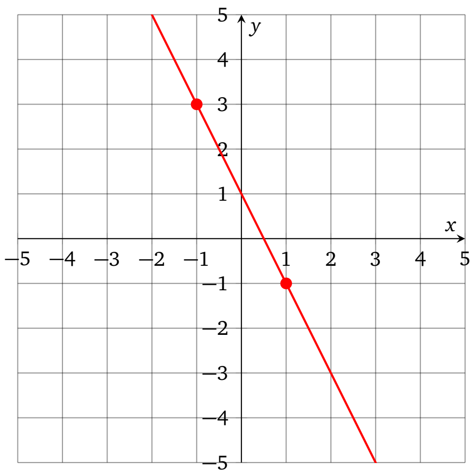

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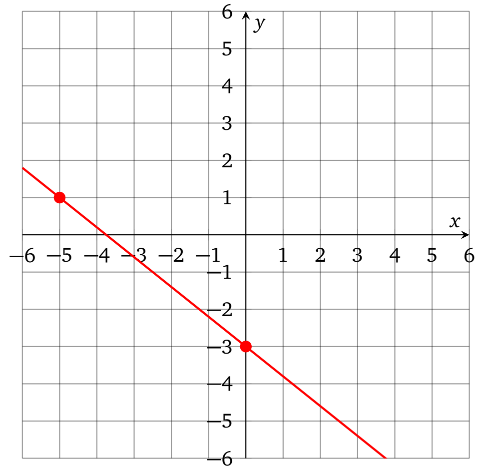

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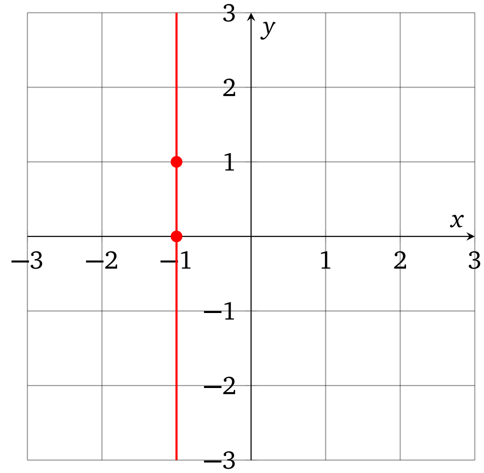

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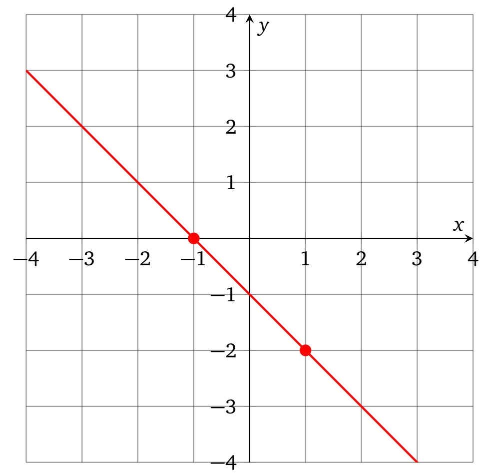

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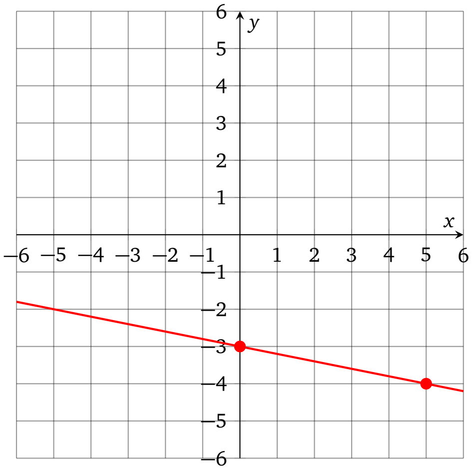
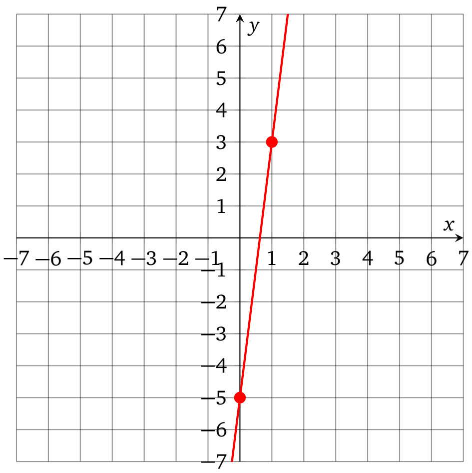

x
y
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

15.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y
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7 6 5 4 3 2 1 −1 −2 −3 −4 −5 −6 −7

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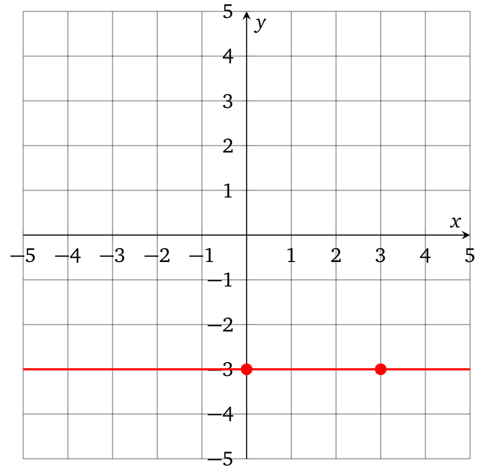
x
y
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5 4 3 2 1 −1 −2 −3 −4 −5

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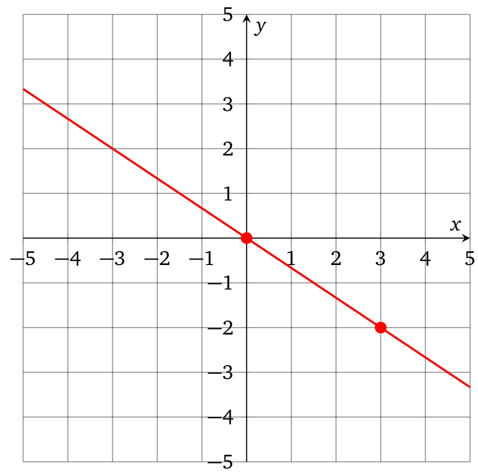

x
y
−5 −4 −3 −2 −1 1 2 3 4 5
5 4 3 2 1 −1 −2 −3 −4 −5

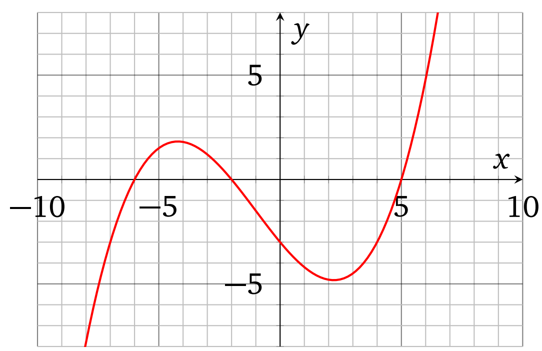
3. Intercepts

3.1. x and y-intercepts.

DEFINITION 3.1.1. For any equation in variables x and y we define *x-intercepts* as the points where the graph of the equation meets the x-axis. In other words, x-intercepts are all the points $(a, 0)$ which solve the given equation.

Similarly, we define *y-intercepts* as the points where the graph of the equation meets the y-axis. In other words, y-intercepts are all the points $(0, b)$ which solve the given equation.

EXAMPLE 3.1.1. Find the intercepts for the pictured graph:



ANSWER: There are three x-intercepts: $(-6, 0)$, $(-2, 0)$, and $(5, 0)$. There is one y-intercept: $(0, -3)$.

3.2. Finding Intercepts.

THEOREM 3.2.1. The most straightforward way to find x-intercepts from an equation is to replace y by 0 and then solve for x. Similarly, the standard way to find y-intercepts is to replace x by 0 in the equation and then solve for y.

EXAMPLE 3.2.1. Find the intercepts for the equation $4x - 5y = 40$, then use them to plot the line.

SOLUTION: To find x-intercepts, substitute 0 for y and solve for x:

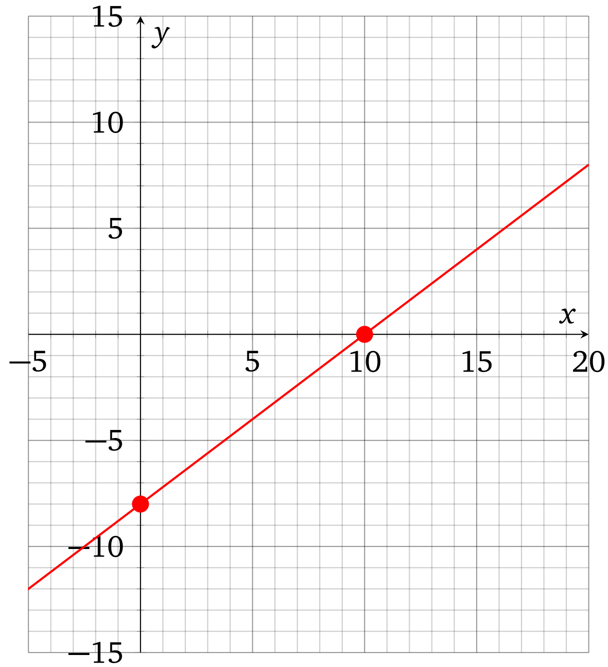
$$\begin{aligned} 4x - 5(0) &= 40 \\ 4x &= 40 \\ x &= 10 \end{aligned}$$

We found the x coordinate of the intercept, and the y coordinate is 0 by definition.

To find y-intercepts, substitute 0 for x and solve for y:

$$\begin{aligned} 4(0) - 5y &= 40 \\ -5y &= 40 \\ y &= -8 \end{aligned}$$

We found the y coordinate of the intercept, and the x coordinate is zero by definition. Now we plot the x-intercept $(10, 0)$ and the y-intercept $(0, -8)$ on the plane, and draw a line through them.



ANSWER: x-intercept $(10, 0)$, y-intercept $(0, -8)$

EXAMPLE 3.2.2. Find the intercepts for the equation $2x = 17$.

SOLUTION: There is no y in this equation, so it is a vertical line, and the only intercept is an x-intercept. We can still follow the steps, though. To find the x-intercept, we solve for x and get $x = 8.5$. To find the y-intercept, we need to plug in 0 for x and solve for y, but the equation $0 = 17$ has no solutions, so there are no y-intercepts.

ANSWER: x-intercept $(8.5, 0)$, y-intercept: none

EXAMPLE 3.2.3. Find the intercepts for the equation $-3y + 6 = 0$.

SOLUTION: There is no x in this equation, so it's a vertical line, and the only intercept is a y-intercept. To find the y-intercept, we solve for y and get $y = 2$. To find the x-intercept, we need to plug in 0 for y and solve for x, but the equation $6 = 0$ has no solutions, so there are no x-intercepts.

ANSWER: x-intercept: none, y-intercept $(0, 2)$

Homework 3.3.

Find the coordinates of all x-intercepts and all y-intercepts without graphing the line.

1. $2x - 7y = 14$

2. $3x + 5y = 15$

3. $9x + 2y = 36$

4. $10x + 3y = 60$

5. $x - 3 = 0$

6. $0 = 4 + y$

7. $y = 2x + 6$

8. $y = 3x - 12$

9. $5x = 4y$

10. $-5x - 6y = 120$

11. $4x + y = 10$

12. $3x = 20 + 6y$

Find the intercepts, and plot the line with intercepts clearly shown.

13. $3x + 4y = 12$

14. $2x - y = 6$

15. $x - 2y = 4$

16. $3x - y = 9$

17. $y = 2x + 6$

18. $y = -3x + 5$

19. $3x - 9 = 3y$

20. $4x - 5y = 20$

21. $-5x + 3y = 180$

22. $10x + 7y = 280$

23. $y = 30 - 3x$

24. $40 + y = 5x$

25. $x + 5 = 1$

26. $y = 3$

27. $-4x = 20y + 80$

28. $60 - 20x = 3x$

29. $3y = -x$

30. $2y - 4x = 0$

31. $y = -2$

32. $x = 7$

Homework 3.3 Answers.

1. x-intercept $(7,0)$, y-intercept $(0,-2)$

3. x-intercept $(4,0)$, y-intercept $(0,18)$

5. x-intercept $(3,0)$, y-intercept: none

7. x-intercept $(-3,0)$, y-intercept $(0,6)$

9. x-intercept $(0,0)$, y-intercept $(0,0)$

11. x-intercept $(2.5,0)$, y-intercept $(0,10)$

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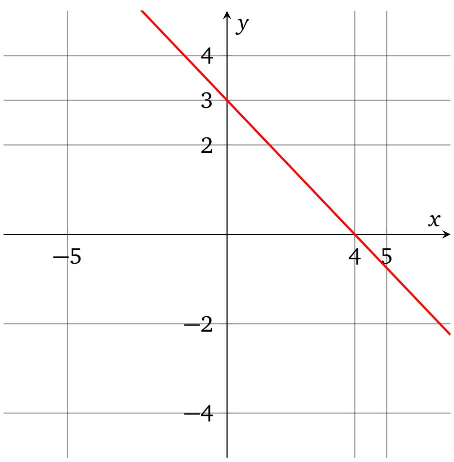

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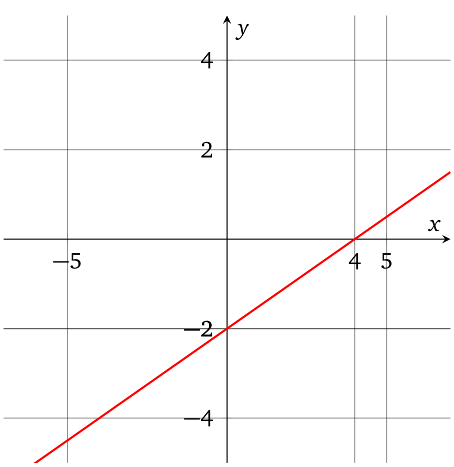

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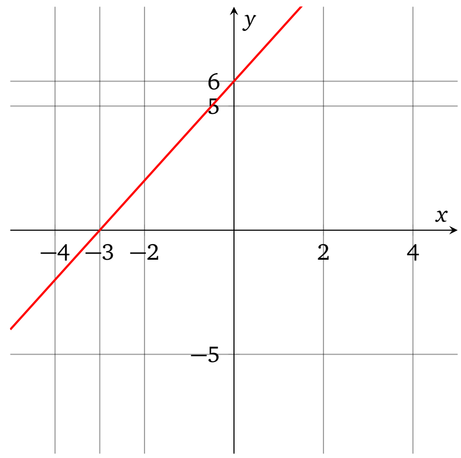

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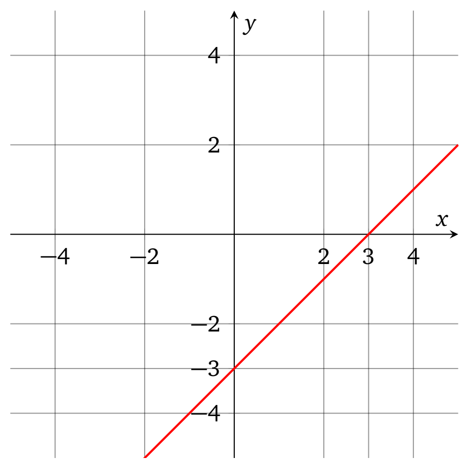

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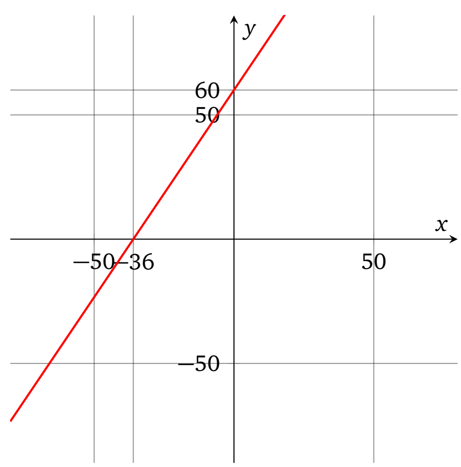

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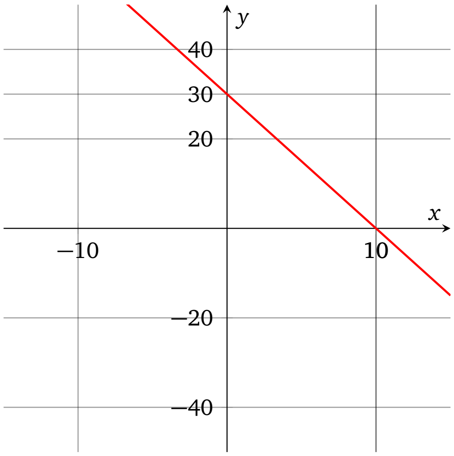

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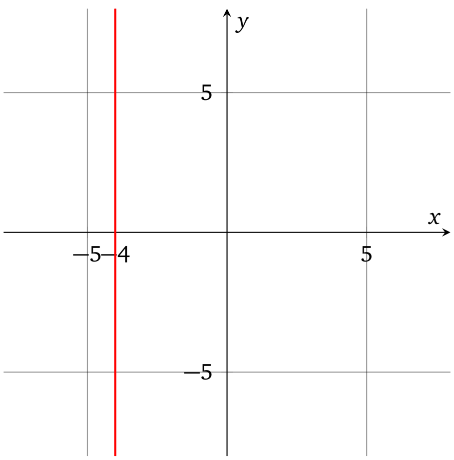

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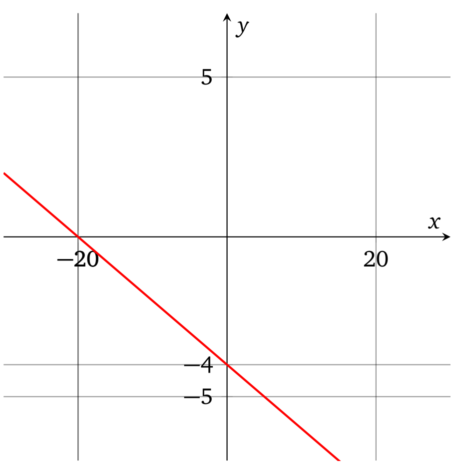

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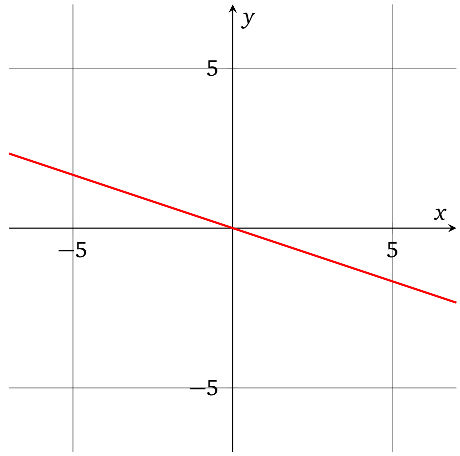

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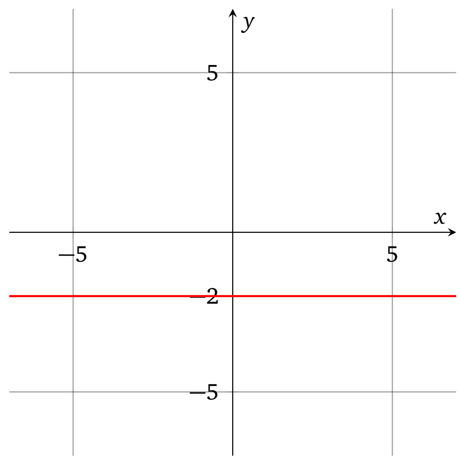
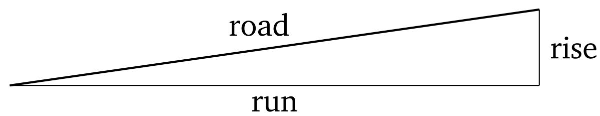


4. Slope

4.1. Slope of a Line. The slope of a line can be understood as a real number which quantifies the how steep the line is. One of the basic applications of the slope is the grade of the road.

DEFINITION 4.1.1. The *grade* of a straight segment of a road is the ratio of the vertical displacement, known as *rise*, and the horizontal displacement, known as *run*. It is traditionally stated in percent, so 9% grade corresponds to the ratio 0.09. In the following illustration, the road with upward slope looks like a hypotenuse of the right triangle with the horizontal leg corresponding to the run, and the vertical leg corresponding to the rise.

$$\text{grade} = \frac{\text{rise}}{\text{run}}$$



EXAMPLE 4.1.1. Find the grade of the road if it rises 400 feet over the horizontal distance of 1500 feet.

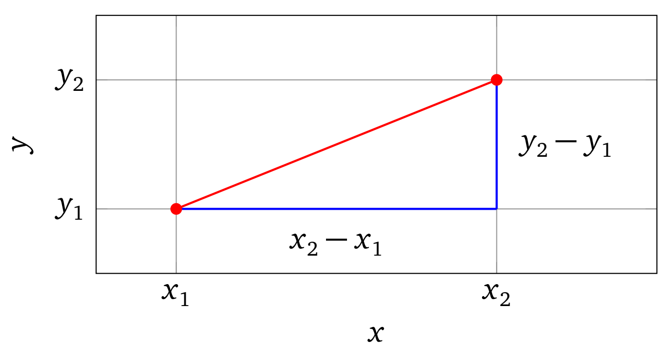
SOLUTION: The grade is $450/1500 = 0.3$, or 30%

ANSWER: 30%

The slope of a line on a coordinate plane is defined similarly, except that the rise carries a sign, and positive slopes correspond to lines going upward and to the right, while negative slopes correspond to lines going downward and to the right.

DEFINITION 4.1.2. The *slope of a segment* with distinct endpoints (x_1, y_1) and (x_2, y_2) is denoted as m and can be computed using the *slope formula*, displayed below. The slope of a vertical segment is left undefined because in that case $x_1 = x_2$, the denominator of the slope formula is zero, and the result of division is undefined.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



DEFINITION 4.1.3. The *slope of a line* is the slope of any segment with two distinct end-points on that line. The slope of a vertical line is left undefined.

The slope of a segment with endpoints (x_1, y_1) and (x_2, y_2) can be understood as the ratio of rise and run, with rise represented by $y_2 - y_1$ and run represented by $x_2 - x_1$. The slope can also be understood as the *change* of y coordinate per unit change of x coordinate, or as the rate of change of y with respect to x.

4.2. Using Slope and Intercept to Plot.

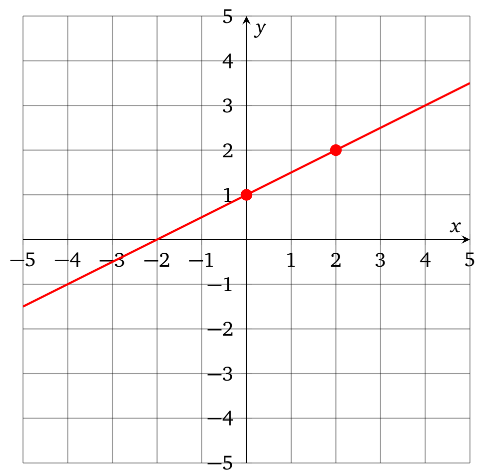
EXAMPLE 4.2.1. Plot the line with slope 1/2 and y-intercept $(0, 1)$

SOLUTION: The slope 1/2 means that

$$\frac{1}{2} = \frac{\text{rise}}{\text{run}}$$

so the line rises 1 unit every time the x coordinate increases by 2 units. We can plot the y-intercept $(0, 1)$, which is a point on the line, then count 2 units to the right, 1 unit up, and plot the point $(2, 2)$. Having found two points, we can complete the graph by drawing a line through them.

ANSWER:



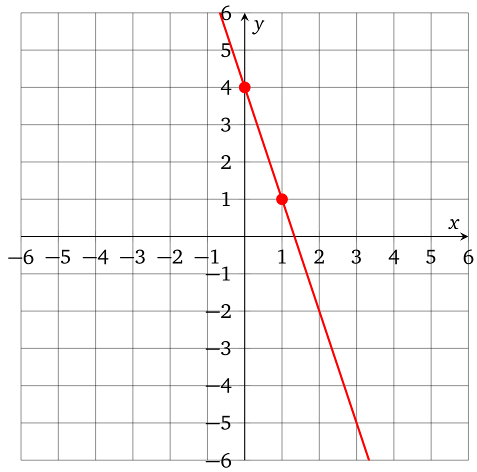
EXAMPLE 4.2.2. Plot the line with slope −3 and y-intercept $(0, 4)$

SOLUTION: The slope −3 can be thought of as a fraction

$$\frac{-3}{1} = \frac{\text{rise}}{\text{run}}$$

so the line falls 3 units every time the x coordinate increases by 1 unit. We can plot the y-intercept $(0, 4)$, which is a point on the line, then count 1 unit to the right, 3 units down, and plot the point $(1, 1)$. Having found two points, we can complete the graph by drawing a line through them.

ANSWER:



Homework 3.4.

Plot the line using the slope and the intercept.

1. Slope $2/3$, y-intercept $(0, -1)$

2. Slope $3/5$, y-intercept $(0, 1)$

3. Slope 0, y-intercept $(0, 1)$

4. Slope 0, y-intercept $(0, -5)$

5. Slope 3, y-intercept $(0, 4)$

6. Slope -3, y-intercept $(0, 2)$

7. Slope -2, y-intercept $(0, -3)$

8. Slope 2, y-intercept $(0, 0)$

9. Slope undefined, x-intercept $(-3, 0)$

10. Slope undefined, x-intercept $(4, 0)$

11. Slope $-4/5$, y-intercept $(0, 6)$

12. Slope $-1/3$, y-intercept $(0, 5)$

13. Slope $1/2$, y-intercept $(0, 0)$

14. Slope $5/3$, y-intercept $(0, -2)$

Find the slope of a segment with given endpoints.

15. $(2, 10)$, $(3, 15)$

16. $(3, 4)$, $(7, 12)$

17. $(-2, 10)$, $(-2, -15)$

18. $(1, 2)$, $(-6, -14)$

19. $(-15, 10)$, $(16, -7)$

20. $(13, -2)$, $(7, 7)$

21. $(10, 18)$, $(-11, -10)$

22. $(-3, 6)$, $(-20, 13)$

23. $(-16, -14)$, $(11, -14)$

24. $(13, 15)$, $(2, 10)$

25. $(7, -14)$, $(-8, -9)$

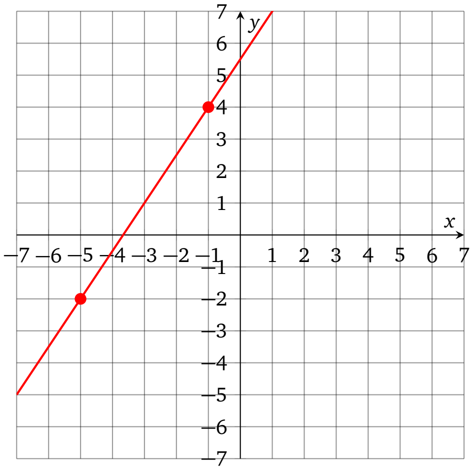
26. $(-18, -5)$, $(14, -3)$

27. $(-5, 7)$, $(-18, 14)$

28. $(19, 15)$, $(5, 11)$

Find the slope of the pictured line.

29.



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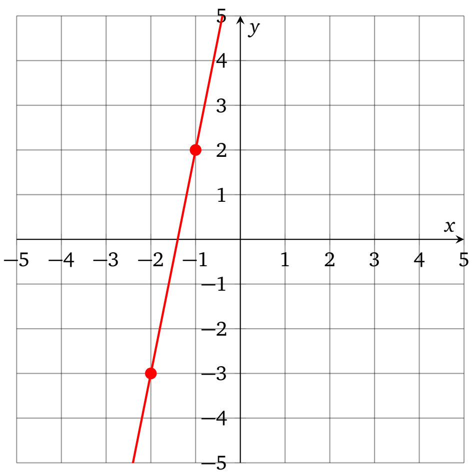

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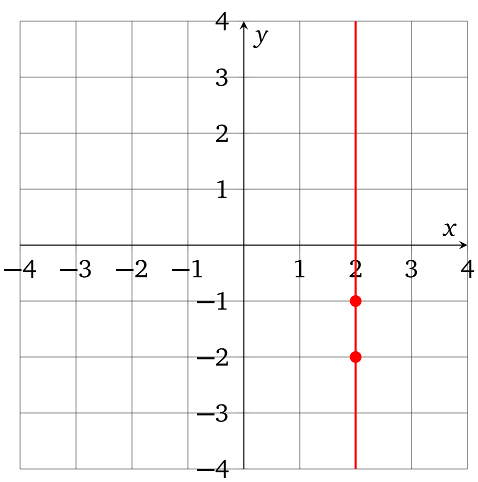

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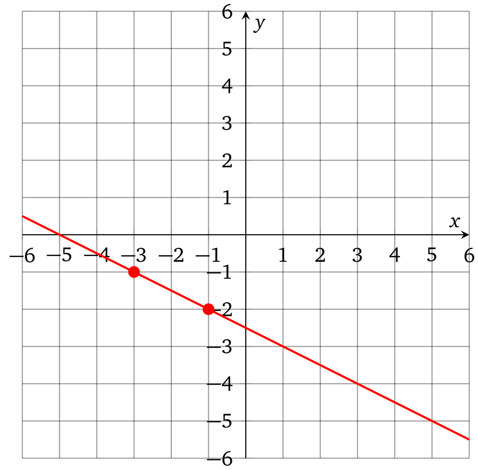

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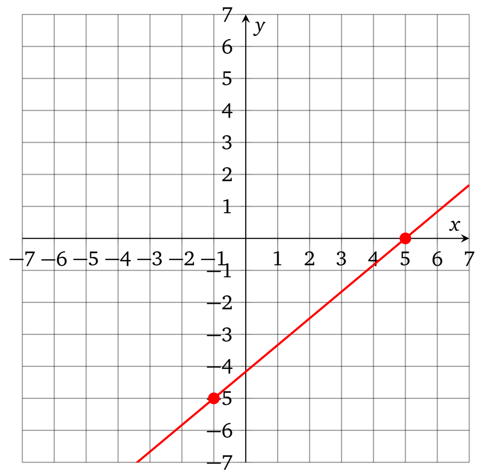

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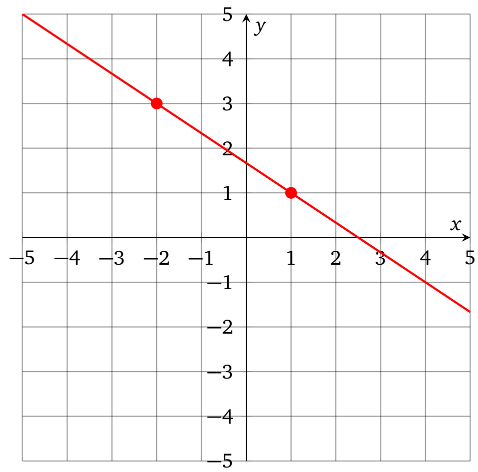

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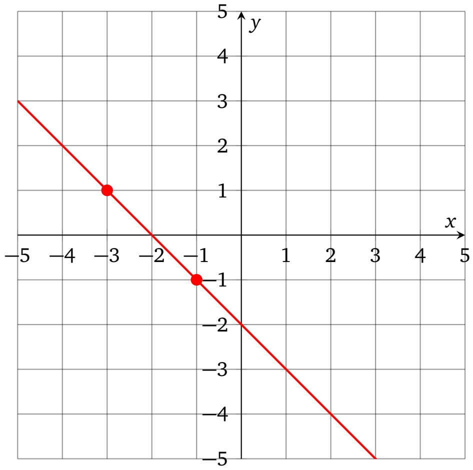

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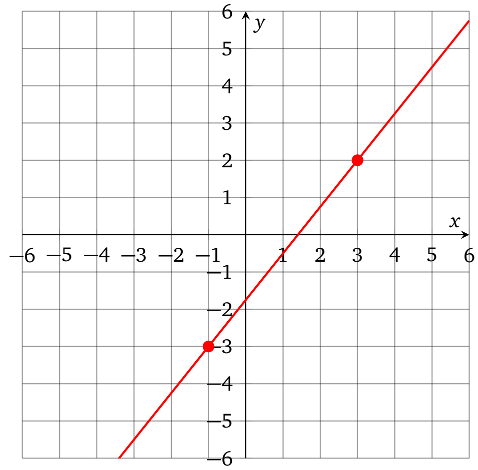

Homework 3.4 Answers.

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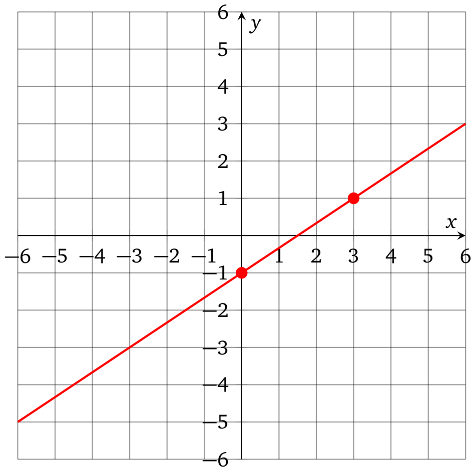

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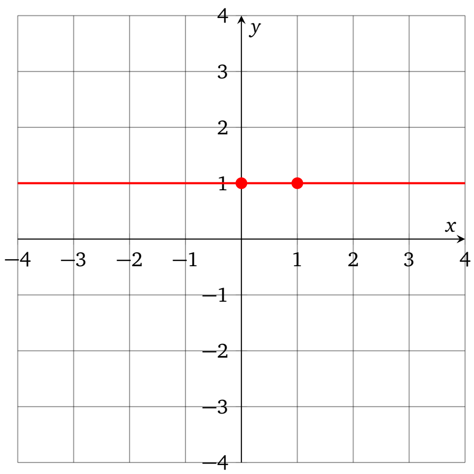

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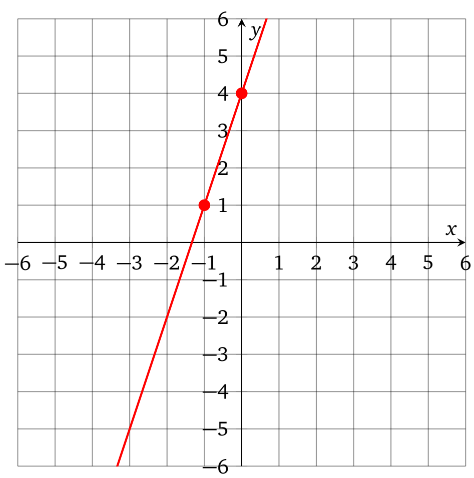

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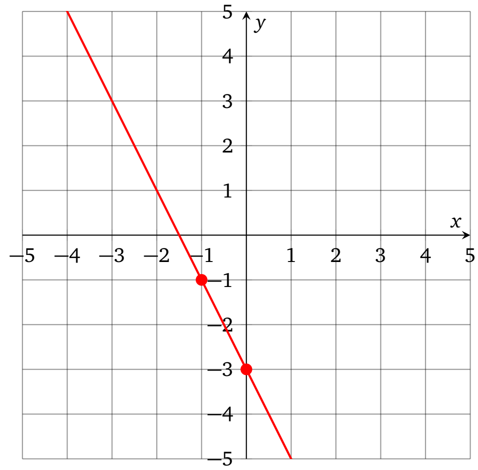

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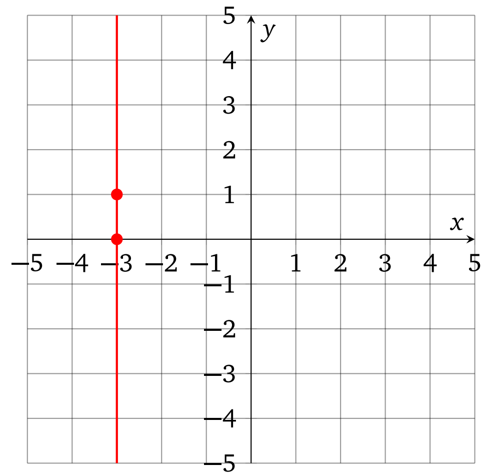

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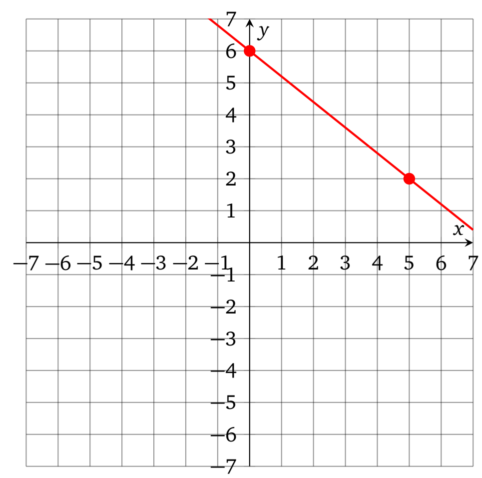

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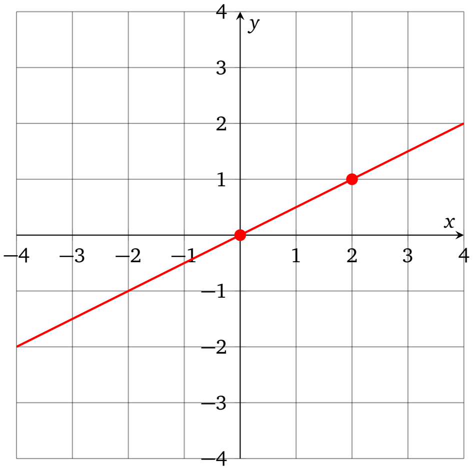


15. 5

17. undefined

19. $-\frac{17}{31}$

21. $\frac{4}{3}$

23. 0

25. $-\frac{1}{3}$

27. $-\frac{7}{13}$

29. $\frac{3}{2}$

31. undefined

33. $\frac{5}{6}$

35. -1

5. Slope-Intercept Form

5.1. Slope-Intercept Form.

DEFINITION 5.1.1. Every non-vertical line with slope m and y-intercept b is a solution set for an equation in the *slope-intercept form*

$$y = mx + b$$

If either m or b is zero, the corresponding term is not shown.

This definition can be thought of as a theorem with an easy proof.

THEOREM 5.1.1. Given any two reals m and b, the slope of the line

$$y = mx + b$$

is m, and the y-intercept is b.

PROOF. We can find the y-intercept of $y = mx + b$ by making x zero and solving for y:

$$\begin{aligned} y &= mx + b \\ y &= m \cdot 0 + b \\ y &= b \end{aligned}$$

To show that the slope is m, take any two points on the line, like $(0, b)$ and $(1, m + b)$, and use the slope formula

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(m + b) - b}{1 - 0} = \frac{m}{1} = m$$

□

BASIC EXAMPLE 5.1.1. Some lines in slope-intercept form:

$y = 3x - 2$ has slope 3 and y-intercept $(0, -2)$

$y = x + 6$ has slope 1 and y-intercept $(0, 6)$

$y = 17$ has slope 0 and y-intercept $(0, 17)$

$y = -x$ has slope -1 and y-intercept $(0, 0)$

BASIC EXAMPLE 5.1.2. Some lines **not** in slope-intercept form:

$2y = 3x + 4$ is not solved for y

$y = -2(x + 5)$ has a product instead of a sum on the right

$y = \frac{\mathbf{4x-5}}{\mathbf{2}}$ has a fraction instead of a sum on the right

EXAMPLE 5.1.1. Find the slope and the y-intercept of the line

$$y = -6x - 14$$

SOLUTION: This equation is already in the slope-intercept form, so -6 is the slope and $(0, -14)$ is the y-intercept.

ANSWER: slope -6, y-intercept $(0, -14)$

To put a linear equation in the slope-intercept form, one can solve for y, remove parentheses, and state the right side in a simplified form.

EXAMPLE 5.1.2. Find the slope, the y-intercept, and the equation in the slope-intercept form for the line

$$x - 2y - 4 = 8$$

SOLUTION: We will actually find the slope-intercept form first, which will provide us with everything else. We start by solving the equation of the line for the y variable.

$$\begin{aligned}
x - 2y - 4 &= 8 && \text{isolate the term with } y \text{ on the left} \\
x - 2y - 4 + 4 - x &= 8 + 4 - x && \text{by adding 4 and } -x \text{ on both sides} \\
-2y &= -x + 12 && \text{combined like terms} \\
y &= \frac{-x + 12}{-2} && \text{divided both sides by } -2 \\
y &= \frac{-x}{-2} + \frac{12}{-2} && \text{rewrote as a sum of two terms} \\
y &= \frac{1}{2}x - 6 &&
\end{aligned}$$

This is the slope-intercept form, so $1/2$ is the slope and $(0, -6)$ is the y-intercept.

ANSWER: slope $1/2$, y-intercept $(0, -6)$

EXAMPLE 5.1.3. If possible, find the slope, the y-intercept, and the equation in the slope-intercept form for the line

$$-2y+3=-5$$

SOLUTION: Solve the equation for the y variable:

$$\begin{aligned} -2y+3 &= -5 \\ -2y+3-3 &= -5-3 \\ -2y &= -8 \\ y &= 4 \end{aligned}$$

There is no x, but we can pretend that its coefficient is zero:

$$y=0x+4$$

This is the slope-intercept form, so 0 is the slope and $(0,4)$ is the y-intercept.

ANSWER: slope 0, y-intercept $(0,4)$

EXAMPLE 5.1.4. If possible, find the slope, the y-intercept, and the equation in the slope-intercept form for the line

$$2x=3x+5$$

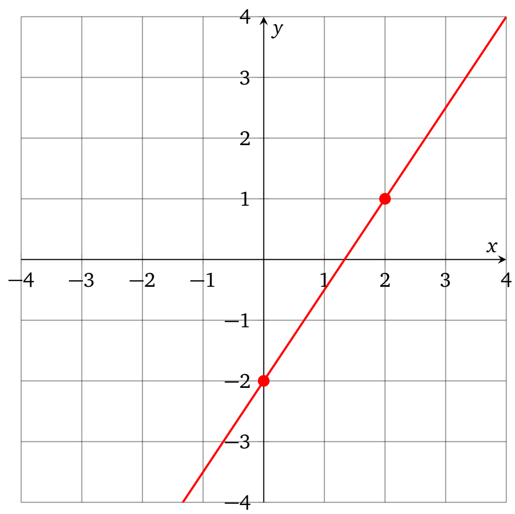
SOLUTION: There is no y, so it is not possible to solve this equation for y. If there are any solutions at all, this is a vertical line, and the slope is undefined. We solve this equation for x to find the x-intercept:

$$\begin{aligned} 2x &= 3x+5 \\ 2x-3x &= 3x+5-3x \\ -x &= 5 && \text{not yet solved for } x \\ x &= -5 && \text{multiplied both sides by } -1 \end{aligned}$$

This is a vertical with the x-intercept $(-5,0)$, so the slope is undefined and there are no y-intercepts.

ANSWER: slope undefined, no y intercepts

EXAMPLE 5.1.5. Determine the equation of the pictured line and state the answer in the slope-intercept form.

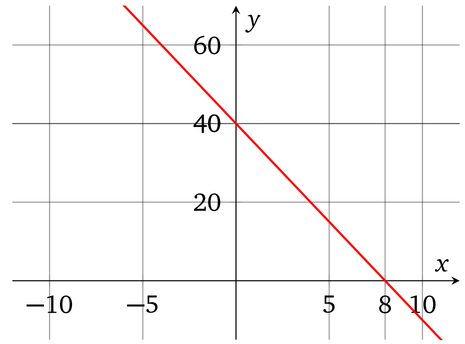


SOLUTION: The y-intercept is $(0,-2)$, so $b=-2$. To find the slope, take any two points on the line, like $(0,-2)$ and $(2,1)$, and use the slope formula with $(0,-2)$ for (x_1,y_1) and $(2,1)$ for (x_2,y_2):

$$m=\frac{y_2-y_1}{x_2-x_2}=\frac{1-(-2)}{2-0}=\frac{3}{2}$$

ANSWER: $y=\frac{3}{2}x-2$

EXAMPLE 5.1.6. Determine the equation of the pictured line and state the answer in the slope-intercept form.



SOLUTION: The y-intercept is $(0,40)$, so $b = 40$. To find the slope, take any two points (x_1, y_1) and (x_2, y_2) on the line, like $(0,40)$ and $(8,0)$, and use the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_2} = \frac{0 - 40}{8 - 0} = -5$$

ANSWER: $y = -5x + 40$

5.2. Parallel Lines. More than 2000 years ago, Euclid provided an axiomatic definition of parallel lines. The classical axioms of geometry imply that given a line and a point not on the line, there exists a unique and distinct *parallel* line passing through that point, and that parallel lines have no points in common. On a coordinate plane, we can provide a consistent definition involving the slope.

DEFINITION 5.2.1. Two distinct lines are *parallel* if their slopes are equal. Just as in classical geometry, a line is not parallel to itself, even though it has the same slope as itself.

BASIC EXAMPLE 5.2.1.

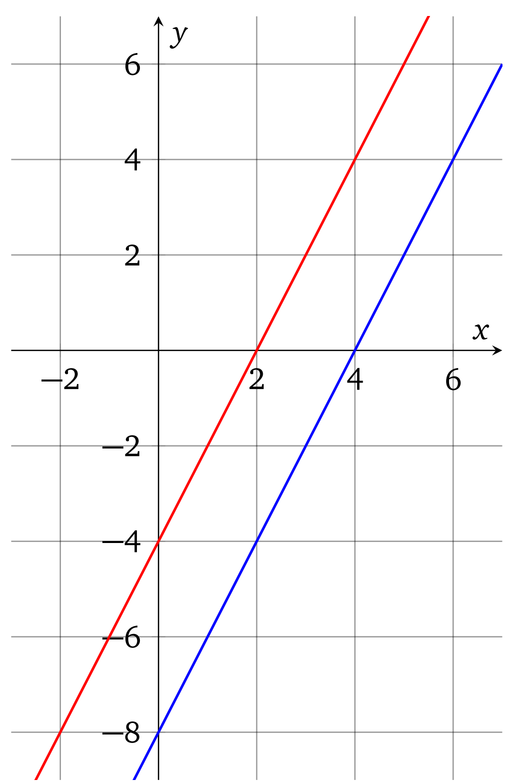
Pictured are two parallel lines:

$y = 2x - 4$

and

$y = 2x - 8$

both with slope $m = 2$.



EXAMPLE 5.2.1. Determine whether the two given lines are parallel.

$$\begin{aligned} y &= 4x - 5 && \text{line 1} \\ 8x - 2y &= 0 && \text{line 2} \end{aligned}$$

SOLUTION: We will find the slopes and compare them. The first equation is in slope-intercept form, so the slope of the first line $m_1 = 4$, and its y-intercept is $(0, -5)$. To find the slope of the second line, we solve its equation for the y variable:

$$\begin{aligned} 8x - 2y &= 0 \\ 8x - 2y - 8x &= 0 - 8x \\ -2y &= -8x \\ \frac{-2y}{-2} &= \frac{-8x}{-2} \\ y &= 4x \end{aligned}$$

This is now also in the slope-intercept form, so the slope of the second line $m_2 = 4$ and its y-intercept is $(0, 0)$. Since the slopes are equal and y-intercepts are different, we conclude that the lines are parallel. (If y-intercepts were the same, we would be looking at two equivalent equations for a single line.)

ANSWER: parallel

EXAMPLE 5.2.2. Determine whether the two given lines are parallel.

$$\begin{aligned} y &= -2x + 4 && \text{line 1} \\ -3y &= 6x - 12 && \text{line 2} \end{aligned}$$

SOLUTION: Divide the equation of the second line by -3 on both sides to solve it for y:

$$\begin{aligned} \frac{-3y}{-3} &= \frac{6x - 12}{-3} \\ y &= \frac{6x}{-3} - \frac{12}{-3} \\ y &= -2x + 4 \end{aligned}$$

So the two equations are equivalent, and the solution sets are the same.

ANSWER: a single line

5.3. Perpendicular Lines. Another useful relationship between two lines is perpendicularity. A geometrical definition requires a right angle between the lines (angle measure of 90° or $\pi/2$ radians), but on a coordinate plane we can provide a consistent definition involving the slope.

DEFINITION 5.3.1. Two lines are *perpendicular* (or *orthogonal*) if their slopes m_1 and m_2 are negative reciprocals of each other:

$$m_1 m_2 = -1$$

Moreover, every vertical line is perpendicular to every horizontal line, even though the slope of a vertical line is undefined.

THEOREM 5.3.1. If neither line is vertical, the definition can be restated in a form that shows the negative reciprocals explicitly. Two lines with non-zero slopes m_1 and m_2 are perpendicular if either

$$m_1 = -\frac{1}{m_2}$$

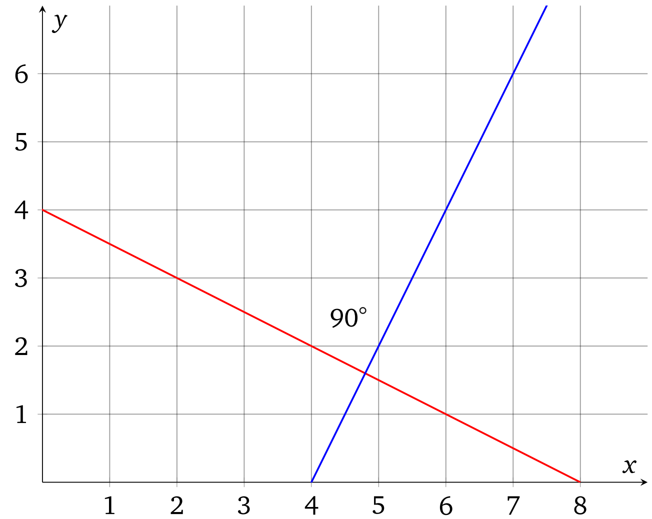
or

$$m_2 = -\frac{1}{m_1}$$

BASIC EXAMPLE 5.3.1. Pictured are two perpendicular lines:

$y = 2x - 8$, with slope $m_1 = 2$ and $y = -\frac{1}{2}x + 4$, with slope $m_2 = -\frac{1}{2}$

Note that $m_1 m_2 = -1$.



EXAMPLE 5.3.1. Determine whether the two given lines are parallel, perpendicular, or neither.

$$\begin{aligned} y &= -3x+8 && \text{line 1} \\ 3y-x &= 4 && \text{line 2} \end{aligned}$$

SOLUTION: We will find the slopes and compare them. The first equation is in slope-intercept form, so $m_1 = -3$. To find the slope of the second line, we solve its equation for the y variable:

$$\begin{aligned} 3y-x &= 4 \\ 3y-x+x &= x+4 \\ 3y &= x+4 \\ \frac{3y}{3} &= \frac{x+4}{3} \\ y &= \frac{1}{3}x+\frac{4}{3} \end{aligned}$$

This is now in the slope-intercept form, so the slope $m_2 = 1/3$. The slopes are not equal, so we check whether they are negative reciprocals of each other:

$$m_1 m_2 = -3 \cdot \frac{1}{3} = -1$$

They are, so the lines are perpendicular.

ANSWER: perpendicular

EXAMPLE 5.3.2. Determine whether the two given lines are parallel, perpendicular, or neither.

$$\begin{aligned} y &= 6 && \text{line 1} \\ x &= -6 && \text{line 2} \end{aligned}$$

SOLUTION: The first line is horizontal and the second one is vertical, so they are perpendicular.

ANSWER: perpendicular

EXAMPLE 5.3.3. Determine whether the two given lines are parallel, perpendicular, or neither.

$$\begin{aligned} 2x + y &= 4 && \text{line 1} \\ x + 2y &= 2 && \text{line 2} \end{aligned}$$

SOLUTION: We will find the slopes and compare them. The first equation is equivalent to

$$y = -2x + 4$$

so $m_1 = -2$. To find the slope of the second line, we solve its equation for the y variable and put it in the slope-intercept form:

$$\begin{aligned} x + 2y &= 2 \\ x + 2y - x &= 2 - x \\ 2y &= -x + 2 \\ \frac{2y}{2} &= \frac{-x + 2}{2} \\ y &= \frac{-x}{2} + \frac{2}{2} && \text{rewrote right side as a sum} \\ y &= -\frac{1}{2}x + 1 && \text{slope-intercept form} \end{aligned}$$

so $m_2 = -1/2$. The slopes are not equal, so the lines are not parallel. Next we check whether they are negative reciprocals of each other:

$$m_1 m_2 = (-2)\left(-\frac{1}{2}\right) = 1$$

The result differs from -1, so the slopes are not negative reciprocals of each other, and the lines are not perpendicular.

ANSWER: neither

Homework 3.5.

Find the slope and the y intercept (if any) from the equation by solving for y and finding an equivalent equation in the slope-intercept form. Do not graph the line.

1. $y = \frac{8}{5}x - 4$

2. $y = -\frac{5}{7}x + 13$

3. $y = -60x$

4. $y = -30x - 14$

5. $5x = 7y$

6. $8x - 9y = 0$

7. $12x - 6y = 9$

8. $2x - 5y = 8$

9. $y - 3 = 5$

10. $y = 3$

11. $x = 4$

12. $1 = 7 - x$

13. $3x + 4y = 12$

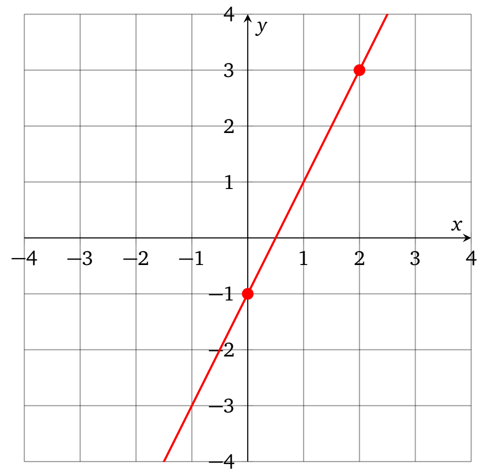
14. $4x + 2y = 8$

15. $-4x + y = 7$

16. $3x - y = 6$

Determine the equation of the pictured line and state the answer in the slope-intercept form.

17.



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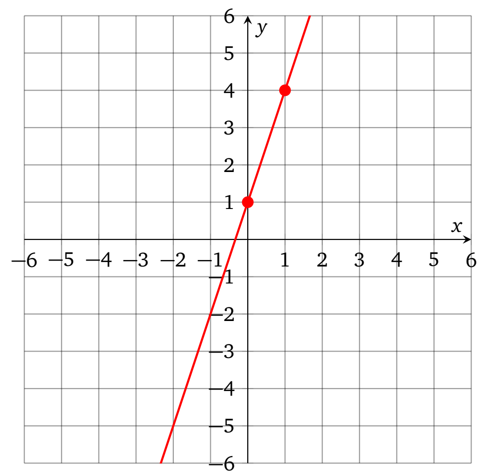

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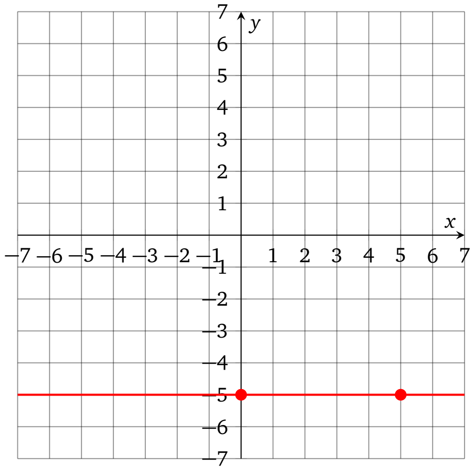

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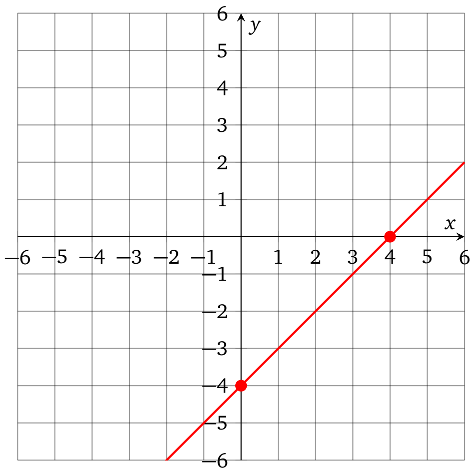

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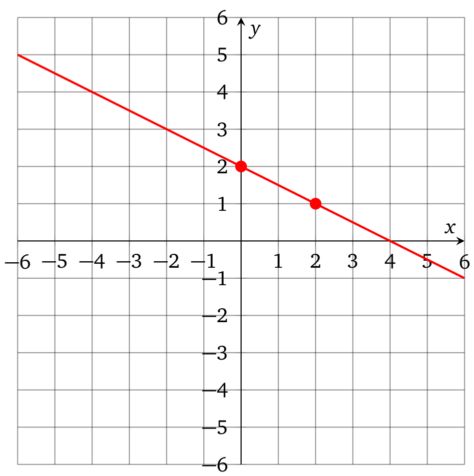

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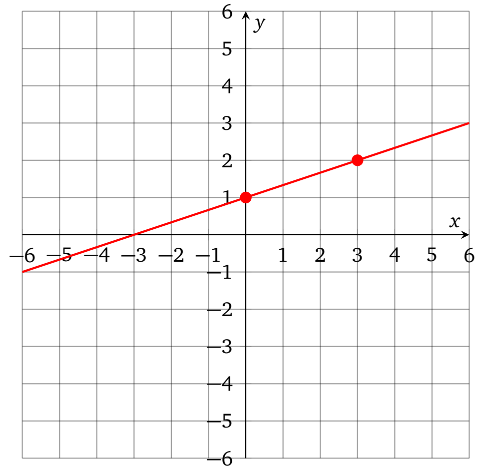

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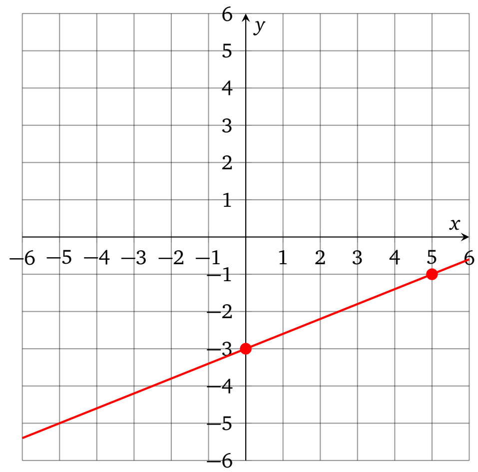

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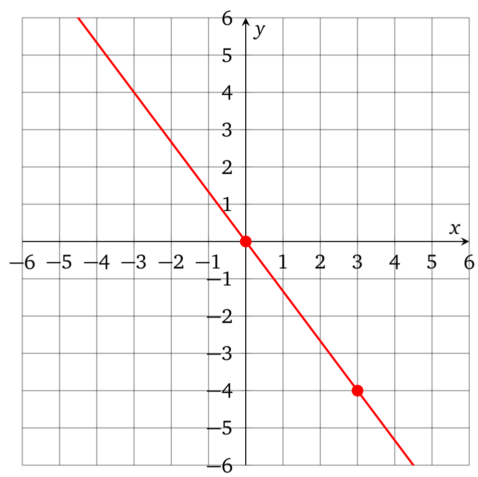


For each pair of lines, determine whether they are parallel, perpendicular, or neither.

25.

$$\begin{aligned} y &= \frac{3}{4}x - 3 \\ 3x - 4y &= -2 \end{aligned}$$

26.

$$\begin{aligned} 8x - 2y &= 7 \\ 3x + 12y &= 9 \end{aligned}$$

27.

$$\begin{aligned} 6x - 4y &= 5 \\ 8x + 12y &= 3 \end{aligned}$$

28.

$$\begin{aligned} 3x - 4y &= 12 \\ y &= \frac{3}{4}x + 3 \end{aligned}$$

29.

$$\begin{aligned} 2x - 4y &= 6 \\ x - 2y &= 3 \end{aligned}$$

30.

$$\begin{aligned} 6x - 3y &= 9 \\ 2x - y &= 3 \end{aligned}$$

31.

$$\begin{aligned} x &= 7 \\ x - 8 &= 0 \end{aligned}$$

32.

$$\begin{aligned} x &= 4 \\ y &= 4 \end{aligned}$$

33.

$$\begin{aligned} x - 4y &= 8 \\ 4x + y &= 2 \end{aligned}$$

34.

$$\begin{aligned} 2x + 5y &= 3 \\ 5x - 2y &= 6 \end{aligned}$$

Homework 3.5 Answers.

1. Slope 8/5, y-intercept $(0,-4)$

3. Slope -60, y-intercept $(0,0)$

5. Slope 5/7, y-intercept $(0,0)$

7. Slope 2, y-intercept $(0,-3/2)$

9. Slope 0, y-intercept $(0,8)$

11. Slope undefined, no y-intercept

13. Slope $-3/4$, y-intercept $(0,3)$

15. Slope 4, y-intercept $(0,7)$

17. $y=2x-1$

19. $y=-5$

21. $y=-\frac{1}{2}x+2$

23. $y=\frac{2}{5}x-3$

25. parallel

27. perpendicular

29. neither (same line)

31. parallel

33. perpendicular

6. Point-Slope Form

6.1. A Point and a Slope Determine a Line.

DEFINITION 6.1.1. Every non-vertical line with slope m and passing through the point (x_1, y_1) is a solution set for an equation in the *point-slope form*

$$y - y_1 = m(x - x_1)$$

This definition can be thought of as a theorem with an easy proof.

THEOREM 6.1.1. Given any real m and any point (x_1, y_1), the line

$$y - y_1 = m(x - x_1)$$

has slope m and passes through the point (x_1, y_1).

PROOF. Substitute x_1 for x and y_1 for y into the equation of the line to check that (x_1, y_1) is a solution:

$$\begin{aligned} y_1 - y_1 &= m(x_1 - x_1) \\ 0 &= m \cdot 0 \\ 0 &= 0 \end{aligned}$$

The equation holds, so the line passes through the point (x_1, y_1). To find the slope, find the slope-intercept form by solving for the y variable:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= mx - mx_1 \\ y - y_1 + y_1 &= mx - mx_1 + y_1 \\ y &= mx - mx_1 + y_1 \end{aligned}$$

This does not look like the slope-intercept form, but rewriting the right side as a sum of two terms helps:

$$y = mx + (-mx_1 + y_1)$$

So $-mx_1 + y_1$ is the y-intercept, while m is the slope. □

EXAMPLE 6.1.1. Find an equation in point-slope form for a line with slope 7, passing through the point $(3,4)$.

SOLUTION: Substitute 7 for m and $(3,4)$ for (x_1,y_1) in the point-slope formula

$$y - y_1 = m(x - x_1)$$

to obtain the answer.

ANSWER: $y - 4 = 7(x - 3)$

EXAMPLE 6.1.2. Find an equation in point-slope form for a line with slope $-3/2$, passing through the point $(-2,5)$.

SOLUTION: After we substitute -2 for x_1 and 5 for y_1 in the point-slope formula we get

$$y - 5 = -\frac{3}{2}(x - (-2))$$

Replacing $-(-2)$ by $+2$ makes the answer look nicer, and is considered point-slope form.

ANSWER: $y - 5 = -\frac{3}{2}(x + 2)$

EXAMPLE 6.1.3. Determine the slope and the coordinates of at least one point on the line

$$y + 1 = -5(x - 8)$$

SOLUTION: This equation is in the point-slope form, so $m = -5$ must be the slope, and $(x_1,y_1) = (8,-1)$ must be a point on the line.

ANSWER: Slope -5, point $(8,-1)$

6.2. Two Points Determine a Line. The very first Euclidean postulate states that two distinct points determine a line, so it is natural to ask for an equation of a line, given two points on the coordinate plane.

EXAMPLE 6.2.1. Find the slope and an equation in the point-slope form for the line passing through the points $(1,-7)$ and $(-4,3)$.

SOLUTION: Let $(x_1, y_1) = (1, -7)$, $(x_2, y_2) = (-4, 3)$, and find the slope using the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-7)}{(-4) - 1} = \frac{10}{-5} = -2$$

Substitute -2 for m and $(1, -7)$ for (x_1, y_1) in the point-slope formula to get the answer:

$$y - (-7) = -2(x - 1)$$

ANSWER: $y + 7 = -2(x - 1)$

EXAMPLE 6.2.2. Find the slope and an equation in the **slope-intercept form** for the line passing through the points $(-12, 3)$ and $(-3, 6)$.

SOLUTION: We will find an equation of this line in the point-slope form first, and then solve it for y to get the slope-intercept form. Let $(x_1, y_1) = (-12, 3)$, $(x_2, y_2) = (-3, 6)$, and find the slope using the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 3}{(-3) - (-12)} = \frac{3}{9} = \frac{1}{3}$$

Substitute $1/3$ for m and $(-12, 3)$ for (x_1, y_1) in the point-slope formula to get the equation in the point-slope form:

$$y - 3 = \frac{1}{3}(x - (-12))$$

Solve the equation for y and put the right side in the slope-intercept form:

$$\begin{aligned} y - 3 &= \frac{1}{3}(x + 12) && \text{point-slope form} \\ y - 3 &= \frac{1}{3}x + \frac{1}{3} \cdot 12 && \text{distributivity} \\ y - 3 &= \frac{1}{3}x + 4 \\ y - 3 + 3 &= \frac{1}{3}x + 4 + 3 && \text{add 3 to both sides} \\ y &= \frac{1}{3}x + 7 && \text{slope-intercept form} \end{aligned}$$

ANSWER: $y = \frac{1}{3}x + 7$

6.3. Using Point and Slope to Plot.

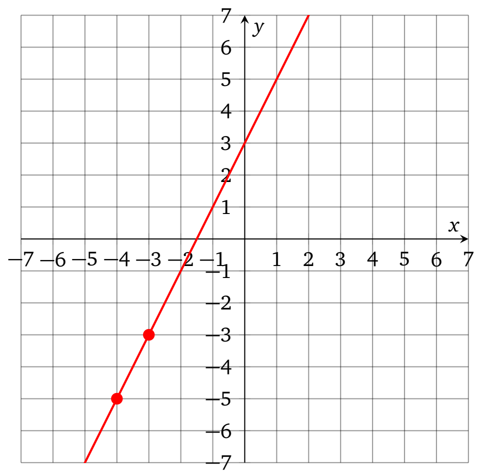
EXAMPLE 6.3.1. Plot the line passing through the point $(-4, -5)$ with slope 2.

SOLUTION: We can think of the slope 2 as

$$\frac{\text{rise}}{\text{run}} = \frac{2}{1}$$

The point $(-4, -5)$ is on the line, and going 2 units up and 1 unit to the right gives us another point on the line $(-3, -3)$.

ANSWER:



Homework 3.6.

Given a slope and a point on the line, state the equation of the line in point-slope form.

1. Slope -1, point $(3,6)$

2. Slope 1, point $(2,8)$

3. Slope -3, point $(-2,-5)$

4. Slope -2, point $(-3,-1)$

5. Slope $7/2$, point $(2,-4)$

6. Slope $-4/5$, point $(-2,6)$

7. Slope $-3/5$, point $(-4,-8)$

8. Slope $1/3$, point $(4,1)$

Determine the slope and the coordinates of at least one point on the line.

9. $y-17=-6(x+1)$

10. $y+4=3(x-2)$

11. $y-7=\frac{1}{2}(x+4)$

12. $y-1=-\frac{2}{3}(x-13)$

13. $y+12=-0.4(x+0.1)$

14. $y-\frac{1}{3}=9\left(x-\frac{1}{4}\right)$

Given two points on the line, find the slope and the equation of the line and state the answer in the slope-intercept form.

15. $(6,8)$ and $(3,5)$

16. $(2,3)$ and $(4,1)$

17. $(-3,1)$ and $(3,5)$

18. $(-3,4)$ and $(3,1)$

19. $(-3,5)$ and $(-1,-3)$

20. $(-4,-1)$ and $(1,9)$

21. $(5,0)$ and $(0,-2)$

22. $(-2,0)$ and $(0,3)$

Graph the line with the given slope and point.

23. Slope $3/4$, point $(1,2)$

24. Slope $2/5$, point $(-3,-4)$

25. Slope $-4/3$, point $(-2,5)$

26. Slope $-3/2$, point $(1,0)$

Graph the equation.

27. $y+2=\frac{2}{3}(x-1)$

28. $y-1=\frac{3}{4}(x+5)$

29. $y-1=-\frac{1}{4}(x-3)$

30. $y-1=-\frac{1}{2}(x-3)$

31. $y+4=\frac{1}{2}(x-1)$

32. $y+2=\frac{1}{3}(x+1)$

Homework 3.6 Answers.

1. $y-6=-1(x-3)$

3. $y+5=-3(x+2)$

5. $y+4=\dfrac{7}{2}(x-2)$

7. $y+8=-\dfrac{3}{5}(x+4)$

9. Slope -6, point $(-1,17)$

11. Slope $\dfrac{1}{2}$, point $(-4,7)$

13. Slope -0.4, point $(-0.1,-12)$

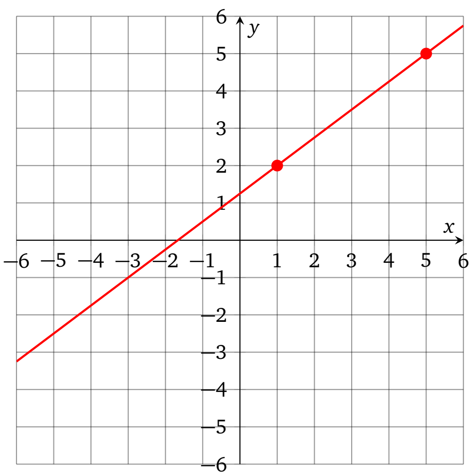
15. Slope 1, $y=x+2$

17. Slope $2/3$, $y=\dfrac{2}{3}x+3$

19. Slope -4, $y=-4x-7$

21. Slope $2/5$, $y=\dfrac{2}{5}x-2$

23.



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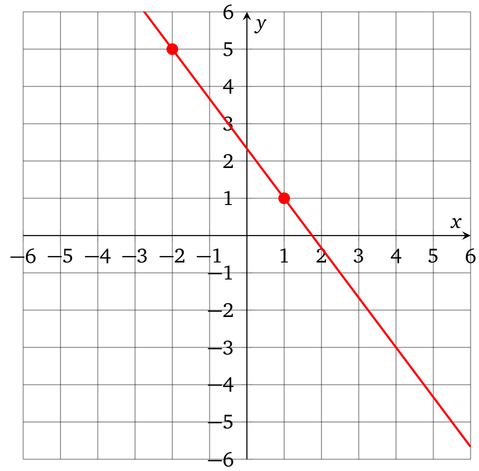

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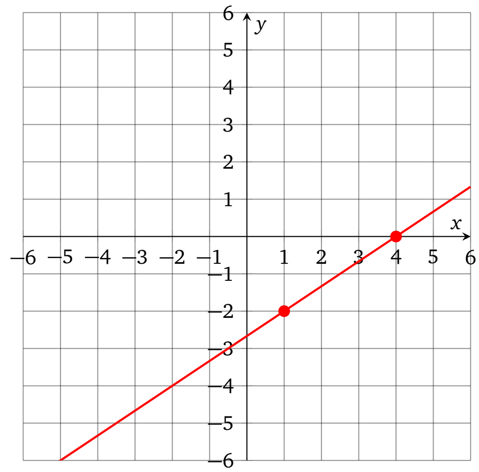

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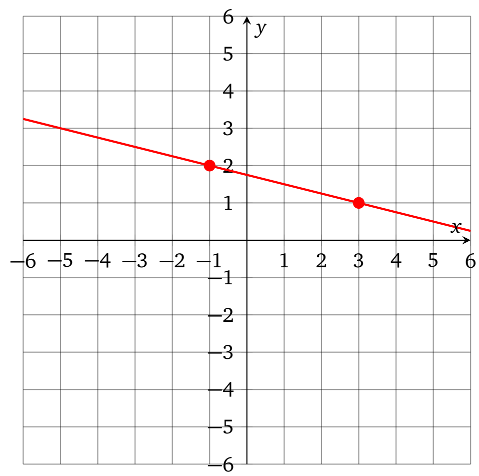

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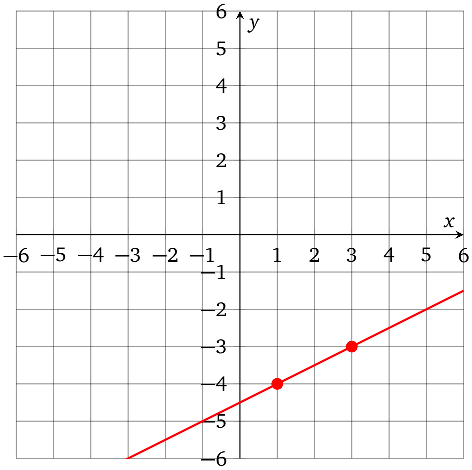


7. Function Notation

7.1. Notation. In modern mathematics, a *function* is a certain kind of collection of ordered pairs. In this text, however, we have no real need for this kind of formalism, so we will only define a specific kind of a function, which is determined by an algebraic expression.

DEFINITION 7.1.1 (Functions defined by expressions). If a formula with variables x and y is solved for y, then we can say that it defines a *function*. We can give this function any name we want, like f, and then f will denote a rule for taking a number x, called *input* or *argument* of the function f, and computing the number y, called *output* or *value* of the function f.

BASIC EXAMPLE 7.1.1. Let the function F be defined by the equation

$$y = x^2 - 2x$$

When the argument $x = 4$, the value of F is

$$(4)^2 - 2(4) = 8$$

When the argument $x = -6$, the value of F is

$$(-6)^2 - 2(-6) = 36 + 12 = 48$$

In other words, finding the value of F amounts to substituting the argument into the formula which defines the function, and computing the result.

DEFINITION 7.1.2. Given a function f, the notation $f(x)$ denotes the value of f corresponding to the argument x. While this looks exactly like a product of variables f and x, the awkward placement of parentheses is usually enough to help the reader realize that f is not a variable, but a name of a function. The parentheses here are not the usual algebraic parentheses, and so they do not obey the distributive property. The only way to get rid of $f(x)$ in an expression is to replace it by the corresponding value of f.

$f(x)$ reads "f of x" or "the value of f at x".

It is very traditional to create functions which share the name with the variable used to define them. For example, a function defined by

$$y = x + 1$$

may also be defined by

$$y(x) = x + 1$$

and then it is also called y. Even so, it is usually easy to determine the meaning of an expression, since function names do not appear in arithmetic expressions without the parentheses and the argument.

EXAMPLE 7.1.1. Let y be a function defined by

$$y(x) = x - 4$$

Find $y(15)$.

SOLUTION: Substitute 15 for x in the expression $x - 4$ to find the value of $y(15)$:

$$y(15) = (15) - 4 = 11$$

ANSWER: 11

EXAMPLE 7.1.2. Let g be a function defined by

$$g(x) = \frac{|x+6|}{x^2}$$

Find $g(-10)$.

SOLUTION: We have to evaluate the expression inside the absolute value bars before we can remove them:

$$\begin{aligned} g(-10) &= \frac{|(-10)+6|}{(-10)^2} \\ &= \frac{|-4|}{100} \\ &= \frac{4}{100} \\ &= 0.04 \end{aligned}$$

ANSWER: 0.04

The absolute value is actually a traditional notation for a function with the following rule: if the argument (or input) is non-negative, then the value of the function is the same as the argument; if the argument is negative, then the value is the opposite of the argument.

EXAMPLE 7.1.3. Let h be a function defined by

$$h(x) = x^3 + 2$$

Find $h(4) + h(5)$.

SOLUTION: We can find values $h(4)$ and $h(5)$ separately and then add them:

$$\begin{aligned} h(4) &= (4)^3 + 2 = 64 + 2 = 66 \\ h(5) &= (5)^3 + 2 = 125 + 2 = 127 \\ h(4) + h(5) &= 66 + 127 = 193 \end{aligned}$$

ANSWER: 193

EXAMPLE 7.1.4. Let f be a function defined by $f(x) = 12 - 5x$. Find $f(0) - f(10)$.

SOLUTION: Another way to simplify this kind of expression is to substitute $f(0)$ and $f(10)$ with expressions right away, but then we have to introduce the substitution parentheses:

$$\begin{aligned} f(0) - f(10) &= (12 - 5(0)) - (12 - 5(10)) \\ &= 12 - (-38) \\ &= 50 \end{aligned}$$

ANSWER: 50

7.2. Applications of Linear Functions.

DEFINITION 7.2.1. A function f is *linear* if it can be defined by a linear expression

$$f(x) = mx + b$$

A graph of this linear function is the graph of the corresponding linear equation

$$y = mx + b$$

DEFINITION 7.2.2. A *linear model* is a linear function which is designed to model a relationship between two quantities in an application.

EXAMPLE 7.2.1. A water pump fills a water tank with water at a constant rate. There are 14 gallons of water in the tank to begin with, and 30 seconds after the pump starts working the amount of water in the tank goes up to 24 gallons. Let x represent the number of seconds the pump was working, and let $y(x)$ represent the amount of water in the tank at the corresponding time.

(1) Find a linear model $y(x)$ for the amount of water in the tank x seconds after the pump starts working.
(2) Use the model to predict the amount of water in the tank 2 minutes after the pump starts working.
(3) Use the model to predict the number of seconds needed to fill the tank with 100 gallons of water.

SOLUTION: When the pump turns on, the timer is at $x = 0$ and the number of gallons in the tank is $y(x) = 14$. After 30 seconds, $x = 30$ and $y(x) = 24$. It may be helpful to represent the two given data points as a table:

x	$y(x)$
0	14
30	24

(1) We need an equation of the line which goes through the points $(0,14)$ and $(30,24)$. The first point happens to be a y-intercept, but we still need to find the slope:

$$m = \frac{24-14}{30-0} = \frac{10}{30} = \frac{1}{3}$$

Using the slope-intercept form $y = mx + b$ we can write the linear model:

$$y(x) = \frac{1}{3}x + 14$$

(2) The variable x measures time in seconds, so we need to convert 2 minutes into 120 seconds before using our linear model:

$$y(120) = \frac{1}{3} \cdot (120) + 14 = 40 + 14 = 54$$

So 2 minutes after the pump starts, there are 54 gallons of water in the tank.

(3) Substitute 100 for $y(x)$ in the equation of the model and solve for x:

$$\begin{aligned} 100 &= \frac{1}{3}x + 14 \\ 100 - 14 &= 13x + 14 - 14 && \text{addition property} \\ 86 &= \frac{1}{3}x && \text{combined like terms} \end{aligned}$$

To finish solving for x we multiply both sides by 3, which is the reciprocal of $1/3$:

$$
\begin{aligned}
(86)\cdot 3 &= \left(\frac{1}{3}x\right)\cdot 3 && \text{multiplication property}\\
258 &= x
\end{aligned}
$$

So the pump has to work for 258 seconds before the tank has 100 gallons of water.

ANSWER:

(1) $y(x) = \dfrac{1}{3}x + 14$
(2) 54 gallons
(3) 258 seconds

EXAMPLE 7.2.2. In year 2000, Brent's private library contained 240 historical manuscripts, and by 2017 the collection grew to contain 461 historical manuscripts. Let x represent the year since 2000 and let $y(x)$ represent the number of manuscripts in the library in that year.

(1) Find a linear model $y(x)$ for the number of manuscripts in a given year.
(2) Use the model to predict the number of manuscripts in year 2024.
(3) Use the model to predict the year when the library obtains the 1000th manuscript.

SOLUTION: Since x represents the year since 2000, x and the year are related by

$$\text{year} = 2000 + x$$

It may be helpful to represent the two given data points as a table:

Year	x	$y(x)$
2000	0	240
2017	17	461

(1) We need an equation of the line which goes through the points $(0, 240)$ and $(17, 461)$. The first point happens to be a y-intercept, but we still need to find the slope:

$$m = \frac{461 - 240}{17 - 0} = \frac{221}{17} = 13$$

Using the slope-intercept form $y = mx + b$ we can write the linear model:

$$y(x) = 13x + 240$$

(2) Year 2024 corresponds to $x = 24$:

$$y(24) = 13 \cdot 24 + 240 = 552$$

So the library will have 552 manuscripts in year 2024.

(3) Substitute 1000 for $y(x)$ in the equation of the model and solve for x:

$$\begin{aligned} 1000 &= 13x + 240 \\ 760 &= 13x \\ 58\tfrac{6}{13} &= x \end{aligned}$$

x is somewhat over 58, so 1000th manuscript will be added sometime during the year 2058.

ANSWER:

(1) $y(x) = 13x + 240$
(2) 552 manuscripts
(3) year 2058

Homework 3.7.

1. Find $f(4)$ if $f(x) = 3x + 1$

2. Find $g(2)$ if $g(t) = t^3 + 1$

3. Find $h(-6)$ if $h(x) = 2 - 3x$

4. Find $p(-3)$ if $p(t) = t^2 + t$

5. Find $F(-4)$ if $F(s) = |1 + s|$

6. Find $G(0)$ if $G(x) = 2(3x - 7)$

7. Find $H(-2)$ if $H(t) = \dfrac{t-4}{t}$

8. Find $K(-5)$ if $K(x) = \dfrac{x^2-1}{6}$

9. Find $P(-1)$ if $P(x) = x^4 - 4x^3$

10. Find $Q(-3)$ if $Q(s) = 2s^3 - s^2$

11. The temperature of the air was 41 °F at sunrise, and 56 °F 30 minutes after the sunrise. Let x represent the number of minutes since sunrise and let $T(x)$ represent the temperature at that time.

(1) Find a linear model $T(x)$ for the air temperature x minutes after the sunrise.
(2) Use the model to predict the temperature 40 minutes after the sunrise.
(3) Use the model to determine the time when the temperature reaches 46 °F.

12. A cup of hot coffee is served at 97 °F, and 5 minutes later it cools down to 89 °F. Let x be the number of minutes since the coffee was served, and let $T(x)$ be the temperature at that time.

(1) Find a linear model $T(x)$ for the coffee temperature x minutes after it was served.
(2) Use the model to predict the temperature 8 minutes after the coffee was served.
(3) Use the model to determine the time when the temperature reaches 74 °F.

13. The population of a small town was 32.6 thousand people in 1990 and it grew to 33.2 thousand people by year 2000. Let x be the number of years since 1990, and let $y(x)$ be the population in thousands of people.

(1) Find a linear model $P(x)$ for the town population in the corresponding year.
(2) Use the model to predict the population in year 2030.
(3) Use the model to determine the time when the population reaches 34.4 thousand people.

14. A racing car accelerates from zero to 60 km/s over 2.5 seconds. Let t be the elapsed time in seconds and let $v(t)$ be the speed of the car in km/s at that time.

(1) Find a linear model $v(t)$ for the speed of the car t seconds after start.
(2) Use the model to predict the speed after 3 seconds.
(3) Use the model to determine the time when the speed reaches 108 km/s.

Homework 3.7 Answers.

1. 13

3. 20

5. 3

7. 3

9. 5

11.

(1) $T(x) = \frac{1}{2}x + 41$
(2) 61 °F
(3) 10 minutes

13.

(1) $P(x) = 0.06x + 32.6$
(2) 35 thousand people
(3) year 2020

Practice Test 3

1. Graph the equation and highlight two distinct points on the line.

$$y = -\frac{1}{2}x + 3$$

2. Find the coordinates of all x-intercepts and all y-intercepts:

$$3x - y = 12$$

3. Find the coordinates of all x-intercepts and all y-intercepts:

$$y = \frac{7}{2}x - \frac{3}{2}$$

4. Plot the line with slope 2 and y-intercept $(0, -4)$.

5. Find the slope of the segment with end-points $(-6, -3)$ and $(-3, 2)$.

6. Find an equivalent equation in slope-intercept form:

$$10x - 4y = 8$$

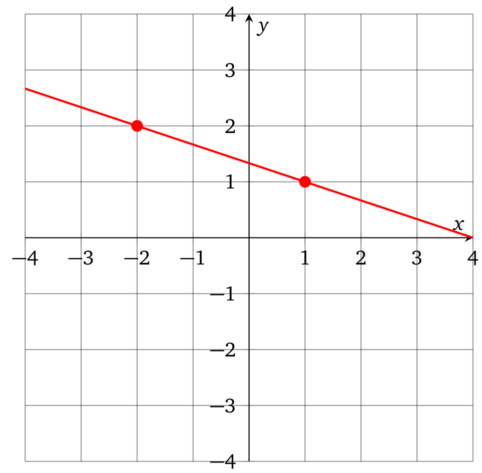
7. Find the slope of the line passing through the points $(3, 0)$ and $(3, -4)$.

8. Determine whether the lines are parallel, perpendicular, or neither:

$$\begin{aligned} y + x &= 4 \\ 2y - 7 &= 2x \end{aligned}$$

9. Find an equation of the line with slope -5 passing through the point $(-2, 3)$ and state it in point-slope form.

10. Determine the equation of the pictured line and state it in the slope-intercept form:



11. Find an equation of the line with slope 0.5 passing through the point $(4, -6)$ and state it in slope-intercept form.

12. Find an equation of the line passing through the points $(-1, 4)$ and $(4, 14)$, and state it in slope-intercept form.

13. Evaluate the function $F(-4)$ if

$$F(x) = 1 - x - x^2$$

14. Find an equation of the line with undefined slope passing through the point $(-7, 7)$.

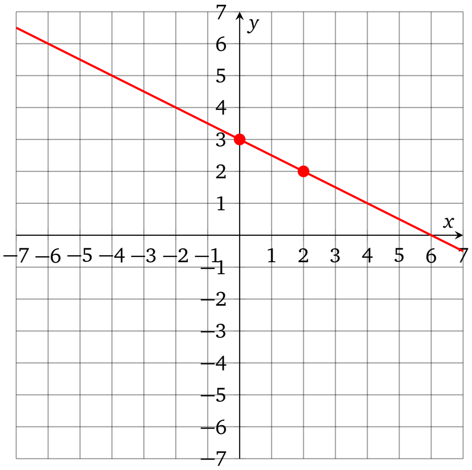
15. Find an equation of the line passing through the point $(5, 0)$ and parallel to the line

$$x + y = 3$$

State the answer in slope-intercept form.

Practice Test 3 Answers.

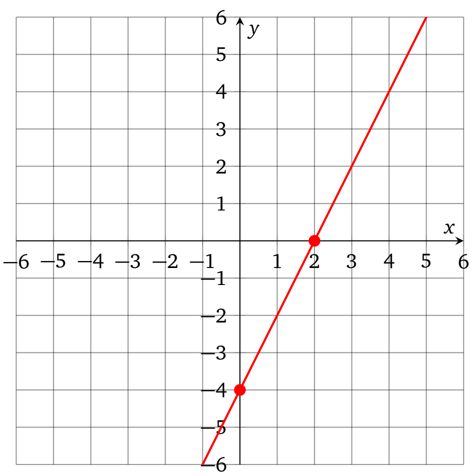
1.



2. x-intercept $(4,0)$, y-intercept $(0,-12)$

3. x-intercept $\left(\frac{3}{7},0\right)$, y-intercept $\left(0,-\frac{3}{2}\right)$

4.



5. $5/3$

6. $y=\frac{5}{2}x-2$

7. undefined

8. perpendicular

9. $y-3=-5(x+2)$

10. $y=-\frac{1}{3}x+\frac{4}{3}$

11. $y=0.5x-8$

12. $y=2x+6$

13. -11

14. $x=-7$

15. $y=-x+5$

CHAPTER 4

Linear Systems

1. Graphing Systems of Equations

1.1. Solution Sets. Recall that solutions to a linear equation in two variables x and y are ordered pairs, which we can plot as points on the Cartesian plane. Recall also that the solution set for any linear equation is a straight line.

DEFINITION 1.1.1. A *system of linear equations in two variables* is two or more linear equations in the same variables x and y, and it is written with a curly brace like this:

$$\begin{cases} x+y &= 5 \\ x-y &= 4 \end{cases}$$

An ordered pair is a *solution* for a system if it is a solution for each and every equation in the system.

EXAMPLE 1.1.1. Determine whether $(2,3)$ is a solution for the system $\begin{cases} x+y &= 5 \\ x-y &= 4 \end{cases}$

SOLUTION: Substitute 2 for x and 3 for y in the first equation:

$$\begin{aligned} x+y &= 5 \\ (2)+(3) &= 5 \\ 5 &= 5 \end{aligned}$$

So it is a solution for the first equation. Now we substitute 2 for x and 3 for y in the second equation:

$$\begin{aligned} x-y &= 4 \\ (2)-(3) &= 4 \\ -1 &= 4 \end{aligned}$$

So it is not a solution for the second equation, and therefore not a solution for the system.

ANSWER: No

EXAMPLE 1.1.2. Determine whether $(4.5, 0.5)$ is a solution for the system

$$\begin{cases} x+y &= 5 \\ x-y &= 4 \end{cases}$$

SOLUTION: Substitute 4.5 for x and 0.5 for y in the first equation:

$$\begin{aligned} x+y &= 5 \\ (4.5)+(0.5) &= 5 \\ 5 &= 5 \end{aligned}$$

So it is a solution for the first equation. Now we substitute 4.5 for x and 0.5 for y in the second equation:

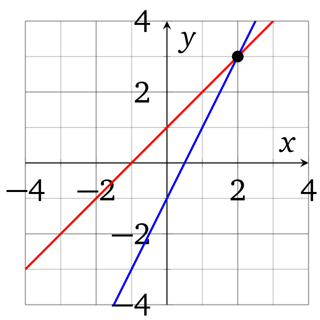
$$\begin{aligned} x-y &= 4 \\ (4.5)-(0.5) &= 4 \\ 4 &= 4 \end{aligned}$$

So it is a solution for the second equation, and therefore a solution for the system.

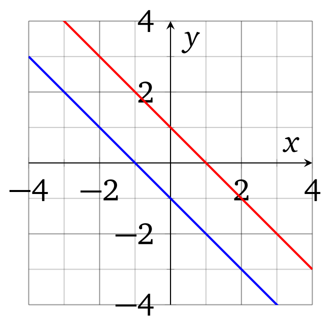
ANSWER: Yes

Knowing that solution sets of individual linear equations look like lines, we can try to figure out the possibilities for the solution sets for systems of two equations. Ordered pairs which satisfy both equations will look like points which belong to both graphs. So there are only three major possibilities.

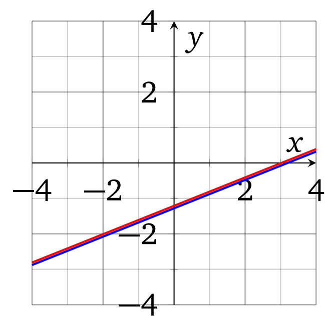
Two lines intersect at a point, yielding a single solution.



Two lines are parallel, yielding an empty solution set.



Two lines coincide, yielding infinitely many solutions

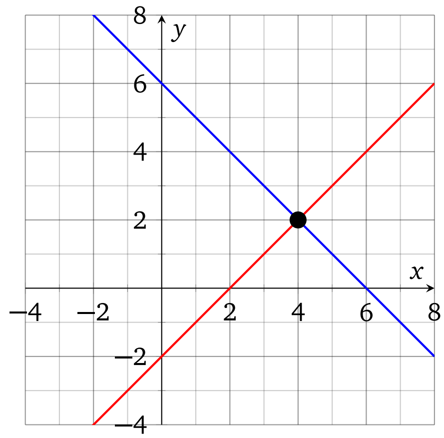


1.2. Solving Systems by Graphing.

EXAMPLE 1.2.1. Solve the system by graphing:

$$\begin{cases} x - y &= 2 \\ x + y &= 6 \end{cases}$$

SOLUTION: Let us plot both lines on the same grid. Here it will be convenient to use intercepts for graphing. The first line $x - y = 2$ has intercepts $(2, 0)$ and $(0, -2)$. The second line $x + y = 6$ has intercepts $(6, 0)$ and $(0, 6)$.



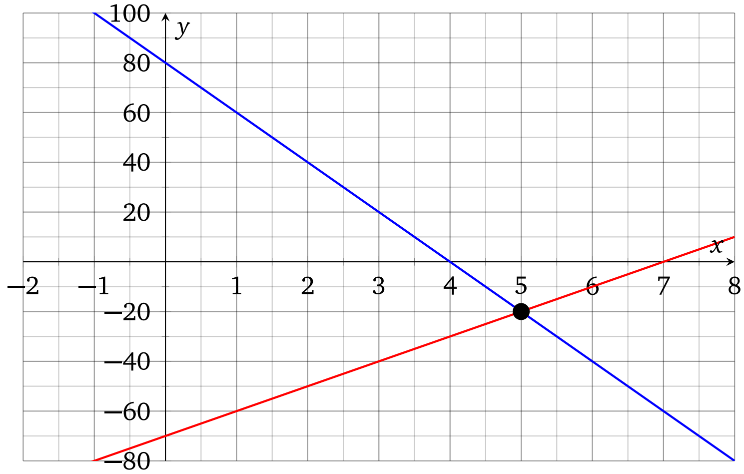
The lines seem to intersect precisely at $(4, 2)$. We can check the solution by making sure $x = 4$ and $y = 2$ satisfy both equations:

$$\begin{aligned} (4) - (2) &= 2 \\ (4) + (2) &= 6 \end{aligned}$$

ANSWER: $(4, 2)$

EXAMPLE 1.2.2. Solve the system by graphing: $\begin{cases} y &= 10x - 70 \\ y &= -20x + 80 \end{cases}$

SOLUTION: Let us plot both lines on the same grid. Here it will be convenient to use the slope-intercept form for graphing. The first line $y = 10x - 70$ has y-intercept $(0, -70)$ and slope 10, which we can plot as going up 10 units and 1 unit to the right. The second line $y = -20x + 80$ has y-intercept $(0, 80)$ and slope -20, which we can plot by finding another point on the line 20 units down and 1 unit to the right.



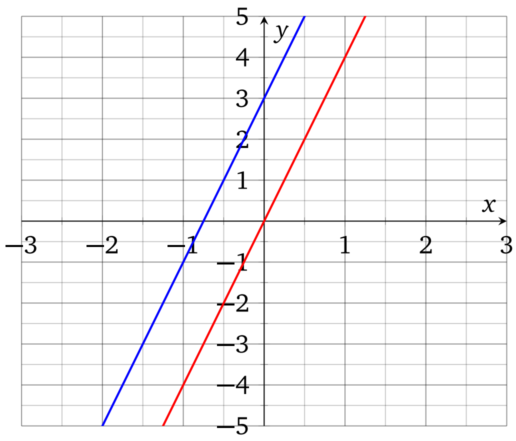
The lines seem to intersect precisely at $(5,-20)$. We can check the solution by making sure $x=4$ and $y=2$ satisfy both equations:

$$\begin{aligned}(-20) &= 10(5)-70\\(-20) &= -20(5)+80\end{aligned}$$

ANSWER: $(5,-20)$

EXAMPLE 1.2.3. Solve the system by graphing: $\begin{cases} 2y = 8x \\ y = 4x+3 \end{cases}$

SOLUTION: The first line in slope-intercept form is $y=4x$, which is a line with slope 4 passing through the origin. The second line has slope 4 and y-intercept 3.



These lines have the same slope, but different y-intercepts, so they are parallel, there are no intersections, and the solution set is empty.

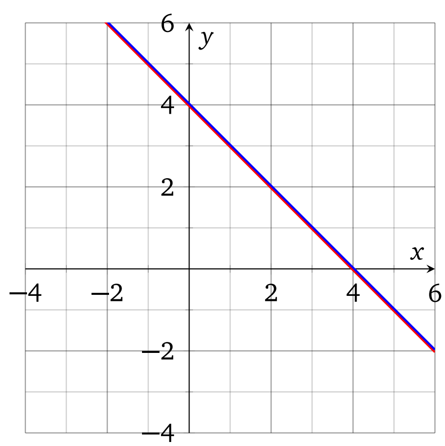
ANSWER: no solutions

EXAMPLE 1.2.4. Solve the system by graphing: $\begin{cases} y+x &= 4 \\ y &= 4-x \end{cases}$

SOLUTION: Solving the first equation for y shows that it is equivalent to the other one:

$$\begin{aligned} y+x &= 4 \\ y+x-x &= 4-x \\ y &= 4-x \end{aligned}$$

Graphing either equation yields the same line:



Every point satisfying the first equation also satisfies the second equation, so there are infinitely many solutions. Formally, we can write down the solution set in the set-builder notation, indicating that any ordered pair which solves one of the equations will also solve the entire system:

$$\{(x,y) \mid y = 4-x\}$$

Without going into specifics, we can also say that there are infinitely many solutions (as many as there are points on a line).

ANSWER: infinitely many solutions

Homework 4.1.

Determine whether the order pair is a solution for the given system of equations.

1. $(1,4)$, $\begin{cases} 5x-2y &= -3 \\ 7x-3y &= -5 \end{cases}$

2. $(-15,20)$, $\begin{cases} 3x+2y &= -5 \\ 4y+5x &= 5 \end{cases}$

3. $(3,2)$, $\begin{cases} 3x-2y &= 0 \\ x+2y &= 15 \end{cases}$

4. $(2,1)$, $\begin{cases} 3x-y &= 5 \\ x+y &= 3 \end{cases}$

5. $(-2,-1)$, $\begin{cases} x-3y &= 1 \\ x+2y &= 0 \end{cases}$

6. $(4,1)$, $\begin{cases} y &= -\frac{1}{2}x+3 \\ y &= \frac{3}{4}x-2 \end{cases}$

Solve the system by graphing.

7. $\begin{cases} y &= -x+1 \\ y &= -5x-3 \end{cases}$

8. $\begin{cases} y &= -3 \\ y &= -x-4 \end{cases}$

9. $\begin{cases} x+3y &= -9 \\ 5x+3y &= 3 \end{cases}$

10. $\begin{cases} y &= 2x+2 \\ y &= -x-4 \end{cases}$

11. $\begin{cases} x-y &= 4 \\ 2x+y &= 2 \end{cases}$

12. $\begin{cases} 6x+y &= -3 \\ x+y &= 2 \end{cases}$

13. $\begin{cases} y &= -\frac{3}{4}x+1 \\ y &= -\frac{3}{4}x+2 \end{cases}$

14. $\begin{cases} y &= 2x-4 \\ y &= \frac{1}{2}x+2 \end{cases}$

15. $\begin{cases} y &= \frac{5}{3}x+4 \\ y &= -\frac{2}{3}x-3 \end{cases}$

16. $\begin{cases} y &= \frac{1}{2}x+4 \\ y &= \frac{1}{2}x+1 \end{cases}$

17. $\begin{cases} y &= -x-2 \\ y &= \frac{2}{3}x+3 \end{cases}$

18. $\begin{cases} x+4y &= -12 \\ 2x+y &= 4 \end{cases}$

19. $\begin{cases} 2x+3y &= -6 \\ 2x+y &= 2 \end{cases}$

20. $\begin{cases} 3x+2y &= 2 \\ 3x+2y &= -6 \end{cases}$

21. $\begin{cases} 2x+y &= -2 \\ x+3y &= 9 \end{cases}$

22. $\begin{cases} x-3 &= 3 \\ 5x+2y &= 8 \end{cases}$

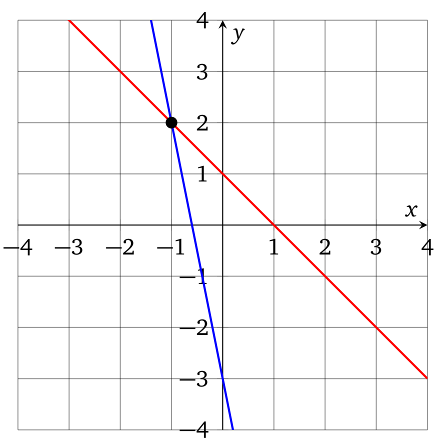
Homework 4.1 Answers.

1. Yes

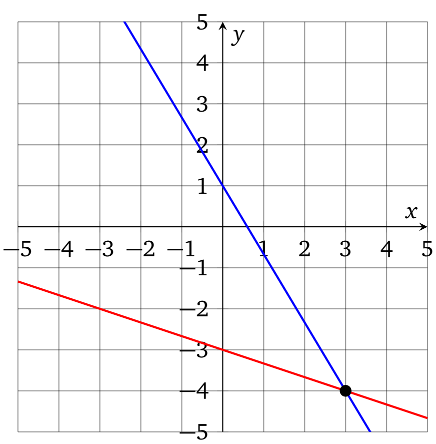
3. No

5. No

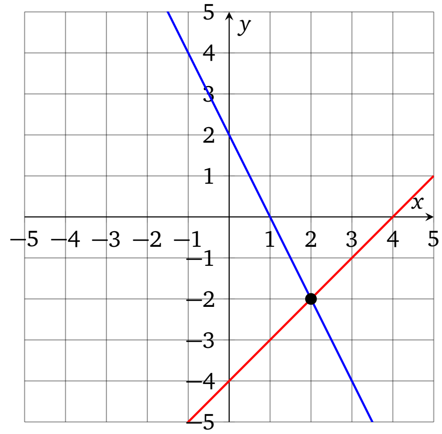
7. $(-1, 2)$



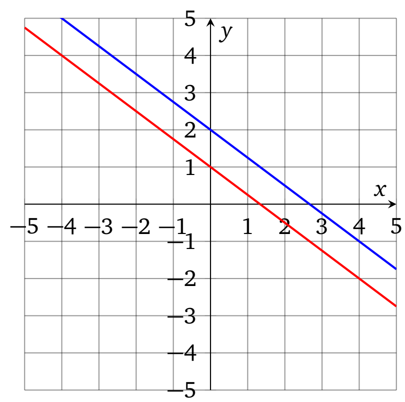
9. $(3, -4)$



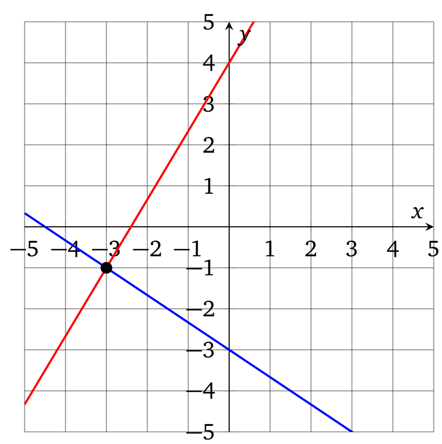
11. $(2, -2)$



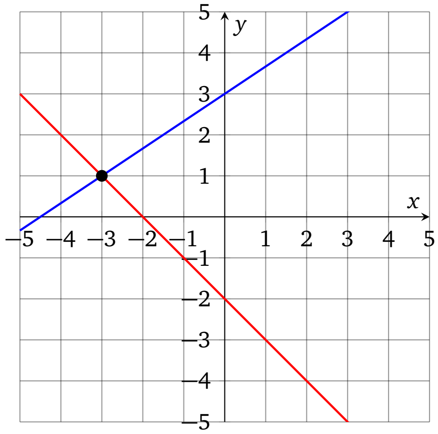
13. no solutions



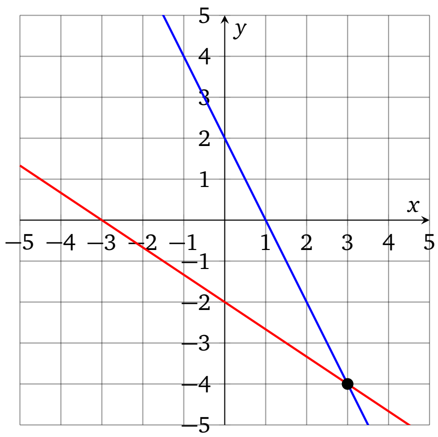
15. $(-3, -1)$



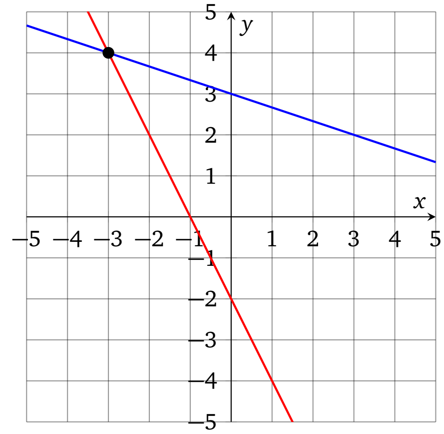
17. $(-3, 1)$



19. $(3, -4)$



21. $(-3, 4)$



2. Substitution

2.1. Solving Systems using Substitution. We can solve systems of linear equations algebraically by eliminating one of the variables using substitution, and then solving a linear equation for the remaining variable.

EXAMPLE 2.1.1. Solve the system $\begin{cases} 7x+y &= 14 \\ y-3x &= -6 \end{cases}$

SOLUTION: Choose a variable to eliminate. Any variable will do, but the ones with low coefficients tend to produce the least tedium. We choose y in the first equation, since it has coefficient 1. Then we find an equivalent equation which is solved for y:

(1) $7x+y = 14$ first equation

(2) $y = 14-7x$ use this equation later, when x is found

Now we can use algebraic substitution to eliminate y from the other equation, by replacing it with $(14-7x)$. This yields a linear equation in one variable x, which we solve by usual means.

$$\begin{aligned} y-3x &= -6 && \text{second equation} \\ (14-7x)-3x &= -6 && \text{substitute } (14-7x) \text{ for y} \\ 14-10x &= -6 && \text{combined like terms} \\ 14-10x-14 &= -6-14 \\ -10x &= -20 \\ \frac{-10x}{-10} &= \frac{-20}{-10} \\ x &= 2 \end{aligned}$$

Finally, we substitute 2 for x in the equation (2), which we solved for y.

$$\begin{aligned} y &= 14-7x \\ y &= 14-7(2) \\ y &= 14-14 \\ y &= 0 \end{aligned}$$

So the only solution is the ordered pair $(2,0)$.

ANSWER: $(2,0)$

This solution technique can be summarized as follows.

THEOREM 2.1.1. To solve a linear system of two equations in two variables,

(1) Solve either equation for either variable to get

$y =$ expression with x or $x =$ expression with y

(2) Substitute the expression into the *other* equation, producing a linear equation in one variable.
(3) Solve the resulting equation to get

$x =$ number or $y =$ number

(4) substitute the number you found into the equation from the first step to find the other coordinate.

When we are dealing with systems which have empty or infinite solution sets, the third step of the procedure will result in an equation which is always false or always true respectively, as the following examples illustrate.

EXAMPLE 2.1.2. Solve the system $\begin{cases} y = x+1 \\ y = x-1 \end{cases}$

SOLUTION: The first equation is already solved for y, so we substitute $(x+1)$ for y in the second equation:

$$\begin{aligned} (x+1) &= x-1 \\ x+1-x &= x-1-x && \text{subtract } x \text{ on both sides} \\ 1 &= -1 \end{aligned}$$

This equation is always false, so no ordered pair (x, y) can ever satisfy the system.

ANSWER: no solutions

EXAMPLE 2.1.3. Solve the system $\begin{cases} x+3y = -10 \\ 6y = -2x-20 \end{cases}$

SOLUTION: We choose to eliminate x from the first equation because it has the most convenient coefficient.

$$\begin{aligned} x+3y &= -10 \\ x &= -3y-10 \end{aligned}$$

Substitute $(-3y-10)$ for x into the other equation:

$$\begin{aligned} 6y &= -2x-20 && \text{the other equation} \\ 6y &= -2(-3y-10)-20 && \text{now we will solve this for } y \\ 6y &= 6y+20-20 \\ 6y &= 6y \end{aligned}$$

We can keep going and subtract $6y$ on both sides to get an equivalent equation $0=0$, but any time we get the same expression on both sides, we can instantly conclude that the linear equation is true for all numbers, which implies that the system has infinitely many solutions.

ANSWER: infinitely many solutions

Homework 4.2.

Solve using substitution.

1. $\begin{cases} y = -3x \\ y = 6x - 9 \end{cases}$

2. $\begin{cases} y = x + 5 \\ y = -2x - 4 \end{cases}$

3. $\begin{cases} x = 2y + 1 \\ 3x - 6y = 2 \end{cases}$

4. $\begin{cases} x = -3y \\ x + 4y = 10 \end{cases}$

5. $\begin{cases} y = -2x - 9 \\ y = 2x - 1 \end{cases}$

6. $\begin{cases} y = -6x + 3 \\ y = 6x + 3 \end{cases}$

7. $\begin{cases} y = 6x - 6 \\ -3x - 3y = -24 \end{cases}$

8. $\begin{cases} -x + 3y = 12 \\ y = 6x + 21 \end{cases}$

9. $\begin{cases} y = 3x - 1 \\ x - \frac{1}{3}y = \frac{1}{3} \end{cases}$

10. $\begin{cases} x - 3y = 7 \\ -4x + 12y = 28 \end{cases}$

11. $\begin{cases} y = -6 \\ 3x - 6y = 30 \end{cases}$

12. $\begin{cases} 6x - 4y = -8 \\ y = -6x + 2 \end{cases}$

13. $\begin{cases} -2x - y = -5 \\ x - 8y = -23 \end{cases}$

14. $\begin{cases} 6x + 4y = 16 \\ -2x + y = -3 \end{cases}$

15. $\begin{cases} -6x + y = 20 \\ -3x - 3y = -18 \end{cases}$

16. $\begin{cases} 7x + 5y = -13 \\ x - 4y = -16 \end{cases}$

17. $\begin{cases} y - 2x = -6 \\ 2y - x = 5 \end{cases}$

18. $\begin{cases} 2x - y = 0 \\ y - 2x = -2 \end{cases}$

19. $\begin{cases} 2x + 3y = -2 \\ 2x - y = 9 \end{cases}$

20. $\begin{cases} y = 2x + 5 \\ -2y = -4x - 10 \end{cases}$

Homework 4.2 Answers.

1. $(1, -3)$

3. $\varnothing$

5. $(-2, -5)$

7. $(2, 6)$

9. infinitely many solutions

11. $(-2, -6)$

13. $(1, 3)$

15. $(-2, 8)$

17. $\left(\frac{17}{3}, \frac{16}{3}\right)$

19. $\left(\frac{25}{8}, -\frac{11}{4}\right)$

3. Elimination

3.1. Equivalent Systems. Another algebraic technique for solving systems is known as *Gaussian elimination* or simply *elimination*, or sometimes *addition*. It relies on two ways of obtaining an equivalent system of equations. We do not need the full power of the Gaussian elimination to solve a system of two equations in two variables, so we present a much simplified version of the technique here.

DEFINITION 3.1.1. Systems of linear equations are *equivalent* if they have the same solution sets.

THEOREM 3.1.1. Given a system of equations, we can replace either of the equations by an equivalent one to obtain an equivalent system. In particular, we can apply the multiplication property to either of the equations. Formally, for any real $m \neq 0$ these two systems are equivalent:

$$\begin{cases} a = b \\ c = d \end{cases} \qquad \begin{cases} a = b \\ m(c) = m(d) \end{cases}$$

THEOREM 3.1.2. Given a system of equations, we can apply the addition property to either of the equations. In particular, we can use the fact that the two sides of the first equation are equal, and add them to the two sides of the other equation. Formally, these two systems are equivalent:

$$\begin{cases} a = b \\ c = d \end{cases} \qquad \begin{cases} a = b \\ c + a = d + b \end{cases}$$

3.2. Eliminating Variable with Addition. In the following examples we will use properties of systems to eliminate one of the variables in one of the equations, solve for that variable, then substitute the value we found into the other equation, and find the other coordinate of the solution.

EXAMPLE 3.2.1. Solve the system using elimination:

$$\begin{cases} 2x + y = 7 \\ 3x - y = 3 \end{cases}$$

SOLUTION: Notice that coefficients of the y variable are opposites of each other. If we add left sides of these equations, then y will cancel (will get *eliminated*) and we will be able to solve for x. So we are going to replace the second equation in the system by a sum of the given equations, by writing that the sum of left sides is equal to the sum of right sides:

$$\begin{cases} 2x+y &= 7 \\ (3x-y)+(2x+y) &= 3+7 \end{cases}$$

Simplifying the second equation yields

$$\begin{aligned} (3x-y)+(2x+y) &= 3+7 \\ 3x+2x-y+y &= 10 \\ 5x &= 10 && \text{combined like terms} \\ x &= 2 \end{aligned}$$

Now we can substitute 2 for x into the first equation and solve for y:

$$\begin{aligned} 2x+y &= 7 \\ 2(2)+y &= 7 \\ 4+y &= 7 \\ y &= 3 \end{aligned}$$

So the only solution is the point $(2,3)$.

ANSWER: $(2,3)$

EXAMPLE 3.2.2. Solve the system using elimination: $\begin{cases} x-4y &= 6 \\ 2x+y &= 12 \end{cases}$

SOLUTION: If we add equations now, neither variable will get eliminated. But we can start by applying the multiplication property so that coefficients for one of the variables become opposites of each other. As before, it is probably easier to eliminate the variable with smaller coefficients, so we will try to get rid of x first. Here we only need to multiply the first equation by -2 on both sides:

$$\begin{cases} -2(x-4y) &= -2(6) \\ 2x+y &= 12 \end{cases}$$

Simplifying the first equation yields an equivalent system:

$$\begin{cases} -2x+8y &= -12 \\ 2x+y &= 12 \end{cases}$$

Now that the coefficients for x are opposites of each other, we can replace the second equation using the addition property for systems:

$$\begin{cases} -2x+8y &= -12 \\ (2x+y)+(-2x+8y) &= (12)+(-12) \end{cases}$$

Simplifying the second equation yields

$$\begin{aligned}(2x+y)+(-2x+8y) &= (12)+(-12)\\ 2x-2x+y+8y &= 0\\ 9y &= 0\\ y &= 0\end{aligned}$$

Now we can substitute $y=0$ into the other original equation $x-4y=6$ and solve for x:

$$\begin{aligned}x-4y &= 6\\ x-4(0) &= 6\\ x &= 6\end{aligned}$$

So the only solution is the point $(6,0)$.

ANSWER: $(6,0)$

EXAMPLE 3.2.3. Solve the system using elimination: $\begin{cases} 2x-5y = 12\\ 3x+2y = -1\end{cases}$

SOLUTION: This is, in a way, a worst case scenario, as far as coefficients are concerned. Before we can eliminate a variable using addition, we need to create a pair of opposite coefficients, and here we will have to multiply both equations. We will opt to eliminate x first because the coefficients seem a bit simpler. In order to do so, we will multiply the first equation by 3 on both sides, and the second equation by -2 on both sides:

$$\begin{cases} 3(2x-5y) = 3(12)\\ -2(3x+2y) = -2(-1)\end{cases}$$

Simplifying both equations yields

$$\begin{cases} 6x-15y = 36\\ -6x-4y = 2\end{cases}$$

Notice that now the coefficients for x are opposites of each other, so we can eliminate x using the addition property:

$$\begin{cases} 6x-15y = 36\\ (-6x-4y)+(6x-15y) = 2+36\end{cases}$$

Simplifying the second equation yields

$$-19y=38$$

And solving it for y gives us

$$y=-2$$

Now we can substitute -2 for y into the original equation $2x-5y=12$ and solve for x:

$$\begin{aligned} 2x-5y &= 12 \\ 2x-5(-2) &= 12 \\ 2x+10 &= 12 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

So the only solution is the point $(1,-2)$.

ANSWER: $(1,-2)$

Homework 4.3.

Solve using elimination.

1. $\begin{cases} 4x+2y &= 0 \\ -4x-9y &= -28 \end{cases}$

2. $\begin{cases} -7x+y &= -10 \\ -9x-y &= -22 \end{cases}$

3. $\begin{cases} -x-2y &= -7 \\ x+2y &= 7 \end{cases}$

4. $\begin{cases} -9x+5y &= -22 \\ 9x-5y &= 13 \end{cases}$

5. $\begin{cases} -6x+9y &= 3 \\ 6x-9y &= -9 \end{cases}$

6. $\begin{cases} 5x-5y &= -15 \\ 5x-5y &= -15 \end{cases}$

7. $\begin{cases} 4x-6y &= -10 \\ 4x-6y &= -14 \end{cases}$

8. $\begin{cases} -3x+3y &= -12 \\ -3x+9y &= -24 \end{cases}$

9. $\begin{cases} -x-5y &= 28 \\ -x+4y &= -17 \end{cases}$

10. $\begin{cases} -10x-5y &= 0 \\ -10x-10y &= -30 \end{cases}$

11. $\begin{cases} 2x-y &= 5 \\ 5x+2y &= -28 \end{cases}$

12. $\begin{cases} -5x+6y &= -17 \\ x-2y &= 5 \end{cases}$

13. $\begin{cases} 10x+6y &= 24 \\ -6x+y &= 4 \end{cases}$

14. $\begin{cases} x+3y &= -1 \\ 10x+6y &= -10 \end{cases}$

15. $\begin{cases} 2x+4y &= 24 \\ 4x-12y &= 8 \end{cases}$

16. $\begin{cases} -6x+4y &= 12 \\ 12x+6y &= 18 \end{cases}$

17. $\begin{cases} 8x-5y &= -9 \\ 3x+5y &= -2 \end{cases}$

18. $\begin{cases} 4x+6y &= -1 \\ x-3y &= 2 \end{cases}$

19. $\begin{cases} x+9y &= 1 \\ 2x-6y &= 10 \end{cases}$

20. $\begin{cases} 2x-15-10y &= 40 \\ 28 &= x-4y \end{cases}$

Homework 4.3 Answers.

1. $(-2,4)$

3. Infinite number of solutions

5. $\varnothing$

7. $\varnothing$

9. $(-3,-5)$

11. $(-2,-9)$

13. $(0,4)$

15. $(8,2)$

17. $(-1,0.2)$

19. $\left(4,-\dfrac{1}{3}\right)$

4. Applications of Systems

4.1. Constant Rate Problems.

EXAMPLE 4.1.1. Suzy pays \$111 for an order of 4 black printer cartridges and 7 color cartridges. When supplies start running low, she goes back to the same store and pays \$126 for an order of 3 black and 10 color cartridges. Find the price of one black cartridge and the price of one color cartridge.

SOLUTION: Let b and c be the prices of one black and one color cartridge respectively, in dollars. We can write an equation for the total price of each order. Since 4 black cartridges cost $4b$ dollars and 7 color cartridges cost $7c$ dollars,

$$4b + 7c = 111$$

Similarly, the second order can be expressed as

$$3b + 10c = 126$$

To solve this system by elimination, we can multiply both sides of the first equation by 3, and both sides of the second equation by -4, so that we can add and eliminate the b variable:

$$\begin{cases} 3(4b + 7c) &= 3(111) \\ -4(3b + 10c) &= -4(126) \end{cases}$$

$$\begin{cases} 12b + 21c &= 333 \\ -12b - 40c &= -504 \end{cases} \qquad \text{simplified both equations}$$

Now the coefficients for b are opposites of each other, and after adding equations we get

$$\begin{aligned} (12b + 21c) + (-12b - 40c) &= 333 + (-504) \\ -19c &= -171 && \text{canceled like terms} \\ c &= 9 && \text{divided both sides by } -19 \end{aligned}$$

Now we substitute 9 for c in the original equation $3b + 10c = 126$ and solve for b:

$$\begin{aligned} 3b + 10(9) &= 126 \\ 3b + 90 &= 126 \\ 3b &= 36 && \text{subtracted 90 on both sides} \\ b &= 12 && \text{divided both sides by 3} \end{aligned}$$

ANSWER:
b and c are prices of black and color cartridges respectively, in dollars

$$\begin{cases} 4b + 7c = 111 \\ 3b + 10c = 126 \end{cases}$$

solution: $b = 12, \quad c = 9$

4.2. Mixing Problems. Mixing problems may assume many forms. Traditional applications revolve around mixing appropriate volumes of various chemical solutions in order to get a solution with a specific concentration. In other applications, we could be "mixing" currency bills of two different denominations or different kinds of coffee or grain.

EXAMPLE 4.2.1. A chemist receives an order for 10 liters of 48% isopropyl solution, that is, a solution consisting of 48 parts of isopropyl alcohol and 52 parts of water by volume. The stockroom only has some 15% solution and some 70% solution. How many liters of each must be mixed in order to fulfill the order?

SOLUTION: As usual, we start by assigning variables to the quantities we need to find. Let x and y be the needed volumes of 15% and 70% solutions respectively, measured in liters.

We will solve this problem by writing down and solving a system of two equations. The first one will say that the total volumes of the two solutions add up to 10 liters. The second one will say that the volumes of just the isopropyl in the two solutions add up to the volume of isopropyl in the final product. Before stating the equations, it may be helpful to construct the following table:

	15% solution	70% solution	48% solution
total volume in liters	x	y	10
volume of isopropyl in liters	$0.15x$	$0.70y$	$0.48 \cdot 10$

Of course, mixing the solutions means that the corresponding volumes add up to the correct totals:

$$\begin{cases} x + y &= 10 \\ 0.15x + 0.7y &= 0.48 \cdot 10 \end{cases}$$

We can solve this system using substitution. Solving the first equation for y yields

$$y = 10 - x \tag{3}$$

We can substitute $(10 - x)$ for y in the second equation to find x:

$$\begin{aligned} 0.15x + 0.7(10 - x) &= 0.48 \cdot 10 \\ 0.15x + 7 - 0.7x &= 4.8 && \text{distributivity} \\ -0.55x + 7 &= 4.8 && \text{combined like terms} \\ -0.55x &= 4.8 - 7 && \text{subtracted 7 on both sides} \\ -0.55x &= -2.2 && \text{combined like terms} \\ x &= 4 && \text{divided both sides by } -0.55 \end{aligned}$$

Now we can substitute 4 for x in the equation (3) and find y:

$$\begin{aligned} y &= 10-x \\ y &= 10-(4) \\ y &= 6 \end{aligned}$$

ANSWER:
x and y are volumes of 15% and 70% solutions respectively, in liters

$$\begin{cases} x+y=10 \\ 0.15x+0.7y=0.48\cdot 10 \end{cases}$$

solution: $x=4, \quad y=6$

EXAMPLE 4.2.2. Stefan has in his coin jar a certain number of quarters and nickels. There are 77 coins in the jar, and together they are worth \$6.05. Find how many quarters and how many nickels are in the jar.

SOLUTION: Let q and n be quantities of quarters and nickels in the jar respectively. The total number of coins is 77, which we can translate as

$$q+n = 77 \qquad \text{first equation}$$

Recall that a quarter is worth \$0.25 and a nickel is worth \$0.05. The total worth of coins is \$6.05, which we can write as

$$0.25q+0.05n = 6.05 \qquad \text{second equation}$$

We can solve this system of two equations by substitution. First we solve the first equation for n, because it has a convenient coefficient 1:

$$q+n = 77 \tag{4}$$

$$n = 77-q \qquad \text{use this later to find } n \tag{5}$$

Now we can substitute $(77-q)$ for n in the second equation and solve for q:

$$\begin{aligned} 0.25q+0.05(77-q) &= 6.05 && \\ 0.25q+3.85-0.05q &= 6.05 && \text{distributivity} \\ 0.2q+3.85 &= 6.05 && \text{combined like terms} \\ 0.2q &= 6.05-3.85 && \text{subtracted 3.85 on both sides} \\ 0.2q &= 2.2 && \text{combined like terms} \\ q &= 11 && \text{divided both sides by 0.2} \end{aligned}$$

Finally we substitute 11 for q in the equation (5) and solve for n:

$$\begin{aligned} n &= 77 - q \\ n &= 77 - 11 \\ n &= 66 \end{aligned}$$

ANSWER:
q and n are the quantities of quarters and nickels respectively

$$\begin{cases} q + n = 77 \\ 0.25q + 0.05n = 6.05 \end{cases}$$

solution: $q = 11, \quad n = 66$

Homework 4.4.

1. A collection of dimes and quarters is worth \$15.25. There are 103 coins in all. How many of each is there?

2. A purse contains \$3.90 made up of dimes and quarters. If there are 21 coins in all, how many dimes and how many quarters are there?

3. There were 429 people at a play. Admission was \$1 each for adults and 75 cents each for children. The receipts were \$372.50. How many children and how many adults attended?

4. There were 200 tickets sold for a women's basketball game. Tickets for students were 50 cents each and for adults 75 cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

5. A total of \$27000 is invested, part of it at 12% and the rest at 13%. The total interest after one year is \$3385. How much was invested at each rate?

6. A total of \$50000 is invested, part of it at 5% and the rest at 7.5%. The total interest after one year is \$3250. How much was invested at each rate?

7. A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is \$2.75. Find the number of each type of coin in the bank.

8. A total of 26 bills are in a cash box. Some of the bills are one dollar bills, and the rest are five dollar bills. The total amount of cash in the box is \$50. Find the number of each type of bill in the cash box.

9. A total of \$9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1030. How much was invested at each rate?

10. A total of \$18000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is \$1248. How much was invested at each rate?

11. At a coin-operated laundromat, using the washer 6 times and then the dryer 6 times costs \$13.50, while using the washer 4 times and then the dryer 8 times costs \$12. Find the cost of one wash and the cost of one drying cycle.

12. In June, Luke worked 150 hours at the regular rate and 10 hours at the overtime rate, for the total paycheck of \$4150. In July, Luke worked 145 hours at the regular rate and 16 hours at the overtime rate, for the total paycheck of \$4265. Find the regular and the overtime hourly rates.

13. A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 lb of tea that is 18% jasmine?

14. How many pounds of coffee that is 40% java beans must be mixed with 80 lb of coffee that is 30% java beans to make a coffee blend that is 32% java beans?

Homework 4.4 Answers.

1.
d and q are quantities of dimes and quarters respectively

$$\begin{cases} 0.1d + 0.25q = 15.25 \\ d + q = 103 \end{cases}$$

solution: $d = 70, \quad q = 33$

3.
a and c are quantities of adults and children respectively

$$\begin{cases} 1a + 0.75c = 372.5 \\ a + c = 429 \end{cases}$$

solution: $a = 203, \quad c = 226$

5.
x and y are dollar amounts invested at 12% and 13% respectively

$$\begin{cases} x + y = 27000 \\ 0.12x + 0.13y = 3385 \end{cases}$$

solution: $x = 12500, \quad y = 14500$

7.
n and d are quantities of nickels and dimes respectively

$$\begin{cases} d = 2n - 10 \\ 0.1d + 0.05n = 2.75 \end{cases}$$

solution: $n = 15, \quad d = 20$

9.
x and y are dollar amounts invested at 10% and 12% respectively

$$\begin{cases} x + y = 9000 \\ 0.1x + 0.12y = 1030 \end{cases}$$

solution: $x = 2500, \quad y = 6500$

11.
w and d are the costs of one washer and one dryer cycle respectively, in dollars

$$\begin{cases} 6w + 6d = 13.5 \\ 4w + 8d = 12 \end{cases}$$

solution: $w = 1.50, \quad d = 0.75$

13.
x and y are weights of 20% and 15% teas in pounds

$$\begin{cases} x + y = 5 \\ 0.2x + 0.15y = 0.18 \cdot 5 \end{cases}$$

solution: $x = 3, \quad y = 2$

5. Multivariate Linear Inequalities

5.1. Solution Sets.

DEFINITION 5.1.1. An ordered pair (a, b) is a *solution* for an inequality in two variables x and y if substituting a for x and b for y makes the inequality true. The *solution set* for an inequality in two variables is a collection of all ordered pairs which solve the inequality.

EXAMPLE 5.1.1. Determine whether $(4, -5)$ a solution for the inequality

$$x + 1 < 2y$$

SOLUTION: Substituting 4 for x and (-5) for y yields

$$\begin{aligned} 4 + 1 &< 2(-5) \\ 5 &< -10 \end{aligned}$$

This is false, so $(4, -5)$ is not a solution.

ANSWER: no

EXAMPLE 5.1.2. Determine whether $(0, 12)$ a solution for the inequality

$$-4x + 2y \geq 1$$

SOLUTION: Substituting 0 for x and 12 for y yields

$$\begin{aligned} -4 \cdot 0 + 2 \cdot 12 &\geq 1 \\ 24 &\geq 1 \end{aligned}$$

This is true, so $(0, 12)$ is a solution.

ANSWER: yes

5.2. Solving Inequalities.

THEOREM 5.2.1. The solution set for a linear inequality in two variables x and y is a half of the coordinate plane. The equation of the line separating the solution set from the rest of the plane can be obtained by replacing the inequality sign by $=$.

A strict inequality ($<$ and $>$) solution set does not contain the points of the line, while a non-strict ($\leq$ and $\geq$) inequality solution set does contain the points on the line.

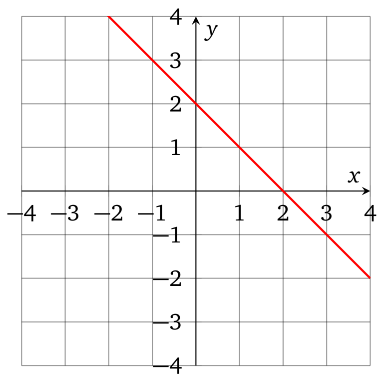
EXAMPLE 5.2.1. Graph the solution set for the inequality

$$x + y \le 2$$

SOLUTION: This is a non-strict inequality, so the points on the line

$$x + y = 2$$

belong to the solution set. We plot the *solid* line first:

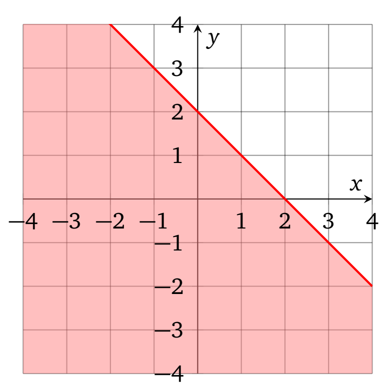


The solution set also includes all points on one side of the line, but which side? One way to find out is by testing any one point not on the line. The point $(0, 0)$, for example, is not on the line. If $(0, 0)$ solves the inequality, then all points on the same side as $(0, 0)$ (which is below the line in this case) belong to the solution set. And if $(0, 0)$ does not solve the inequality, then the points on the other side of the line belong to the solution set. Substituting 0 for both x and y in the inequality yields

$$0 + 0 \le 2$$

This is true, so $(0, 0)$ is a solution, and the entire side of the coordinate plane with $(0, 0)$ consists of solutions, so we graph them all by shading that portion of the plane.

ANSWER:



EXAMPLE 5.2.2. Graph the solution set for the inequality $-2x + y > -1$

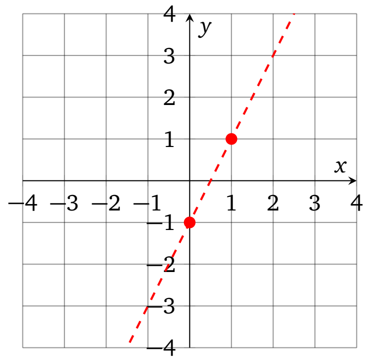
SOLUTION: This is a strict inequality, so the points on the line

$$-2x + y = -1$$

do not belong to the solution set. We are still going to plot the line first, but it will be *dashed* rather than *solid*, to indicate that the line itself is not a part of the solution set. The equation of this line in slope-intercept form is

$$y = 2x - 1$$

So the slope is 2 and the y-intercept is $(0, -1)$.

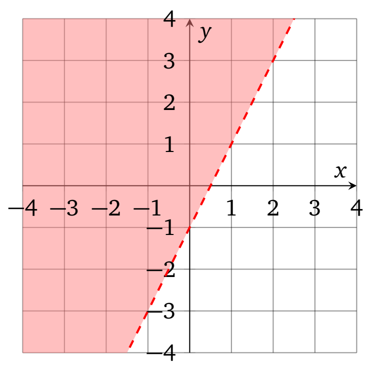


The origin $(0,0)$ is not on the line, so we can use it for locating the solution set. We substitute 0 for both x and y in the inequality to see whether it holds.

$$\begin{aligned} -2 \cdot 0 + 0 &> -1 \\ 0 &> -1 \end{aligned}$$

This is true, so $(0,0)$ is a solution, and the portion of the plane containing $(0,0)$ must be the solution set, so we shade accordingly.

ANSWER:



EXAMPLE 5.2.3. Graph the solution set for the inequality

$$-2x \geq 6$$

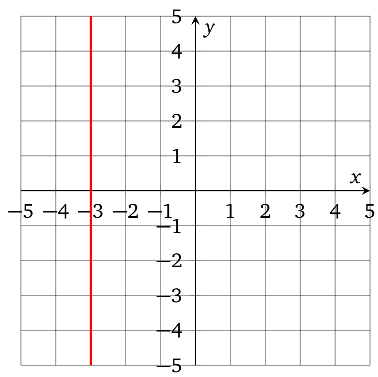
SOLUTION: This is a non-strict inequality, so the points on the line

$$-2x = 6$$

belong to the solution set. This line equation is equivalent to

$$x = -3$$

which is a vertical line with x-intercept $(-3, 0)$. We plot the *solid* line first:

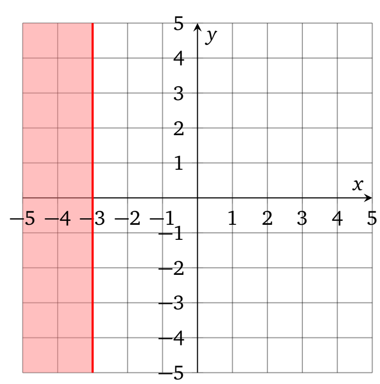


The origin $(0, 0)$ is not on the line, so we can use it for locating the solution set. With 0 for both x and y in the inequality, we get

$$\begin{aligned} -2 \cdot 0 &\geq 6 \\ 0 &\geq 6 \end{aligned}$$

This is false, so the side of the coordinate plane to the right of the line is not a part of the solution set. Hence the points to the left of the line belong to the solution set, and we shade accordingly:

ANSWER:



EXAMPLE 5.2.4. Graph the solution set for the inequality

$$3x < 5y$$

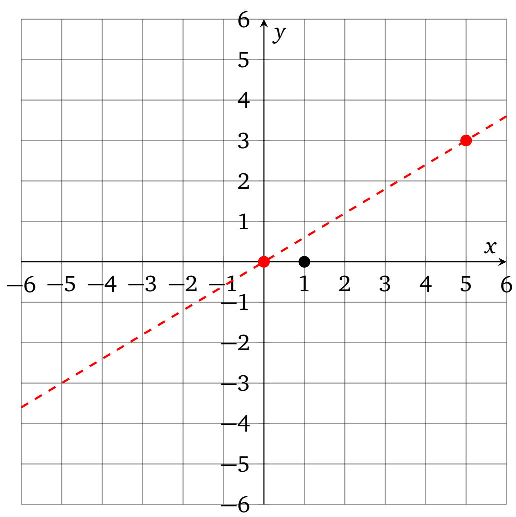
SOLUTION: This is a strict inequality, so we will start by graphing a *dashed* line

$$3x = 5y$$

The slope-intercept form of this equation is

$$y = \frac{3}{5}x$$

so the slope is $3/5$ and the y-intercept is $(0,0)$. We can plot this line by starting at the y-intercept $(0,0)$, and then using the slope to locate another point on the line by going 3 units up and 5 units to the right. This takes us to $(5,3)$, which is another point on this line.

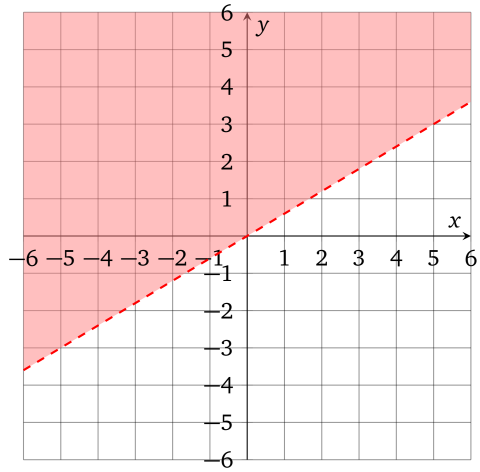


Unlike in previous examples, the origin $(0,0)$ is on the line, so it is useless for locating the solution set. But we can pick any other point for that, as long as it is not on the line. Let us try $(1,0)$, which is just below the line. Substituting 1 for x and 0 for y in the inequality yields

$$\begin{aligned} 3 \cdot 1 &< 5 \cdot 0 \\ 3 &< 0 \end{aligned}$$

This is false, so $(1,0)$ is not a solution, and the portion of the plane below the line is not a part of the solution set. Hence the portion above the line must be the solution set, so we shade accordingly:

ANSWER:



Homework 4.5.

Determine whether the given point is a solution for the given inequality.

1. $(2,6)$
$$2x + y > 10$$

2. $(8,-1)$
$$2x - y \leq -11$$

3. $(-3,2)$
$$x \geq -1$$

4. $(7,-4)$
$$y < 6$$

5. $(0,5)$
$$3x + 3 > y - 3$$

6. $(0,0)$
$$14x < 17y$$

Graph the solution set for the given inequality.

7. $y < x - 2$

8. $y \leq x + 5$

9. $4y \leq 6x$

10. $-3y + 9x > 0$

11. $y \geq x - 1$

12. $y < x + 4$

13. $y > x$

14. $2x \leq y$

15. $x + y \geq 3$

16. $x + y < -1$

17. $2x - 6y \geq 6$

18. $-4x - 8y > 8$

19. $y < -2$

20. $x < 1$

21. $2x - 3y < 12$

22. $3x + 2y \geq -6$

23. $-2x + 5y < -10$

24. $5x + 4y \leq 20$

25. $6y \geq 9x + 18$

26. $5y + 25x \leq -25$

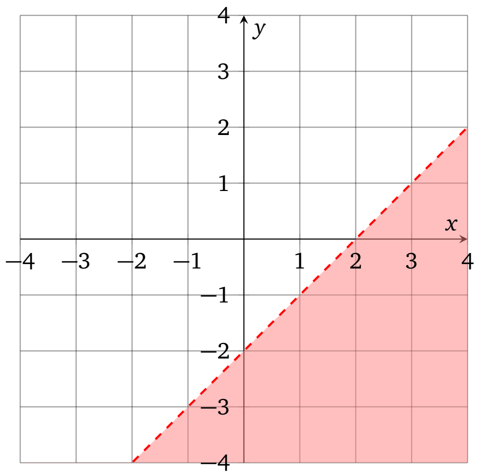
Homework 4.5 Answers.

1. no

3. no

5. yes

7.



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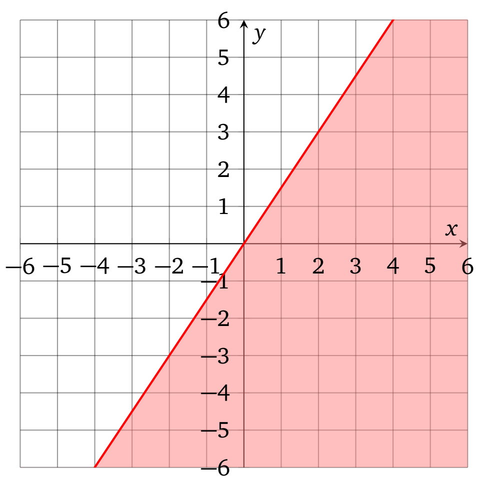

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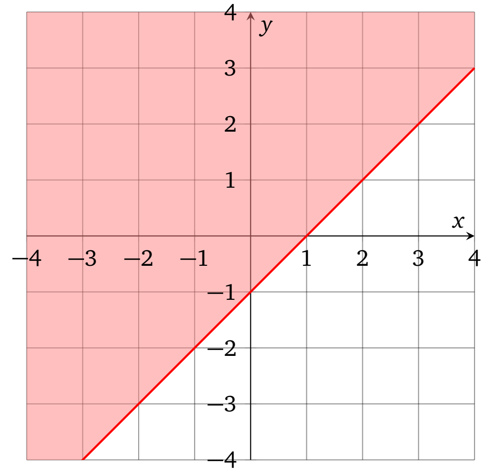

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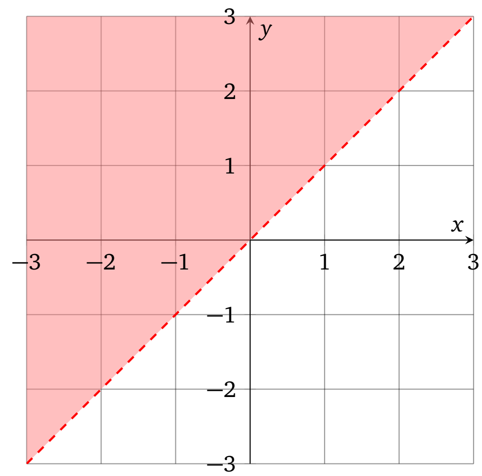

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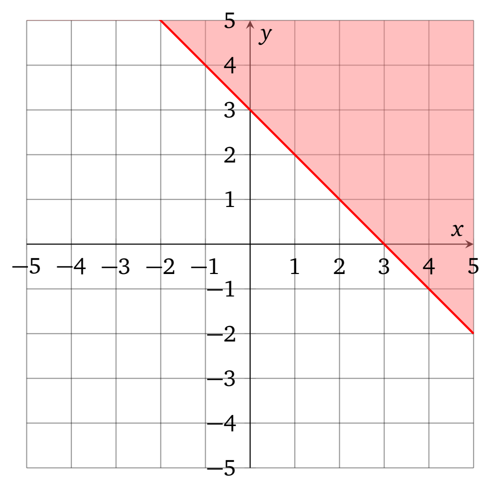

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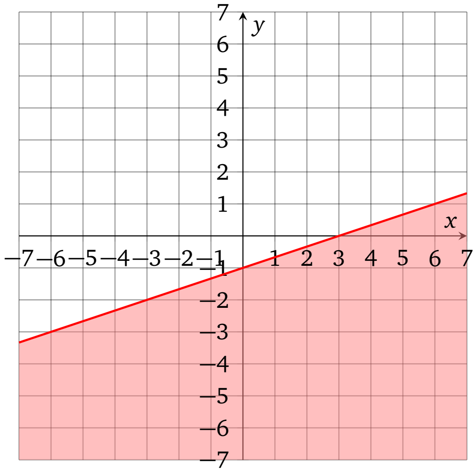
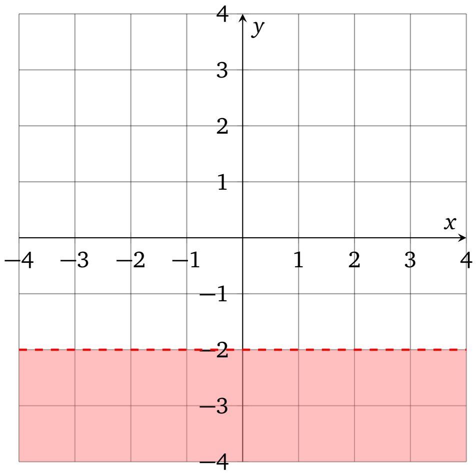
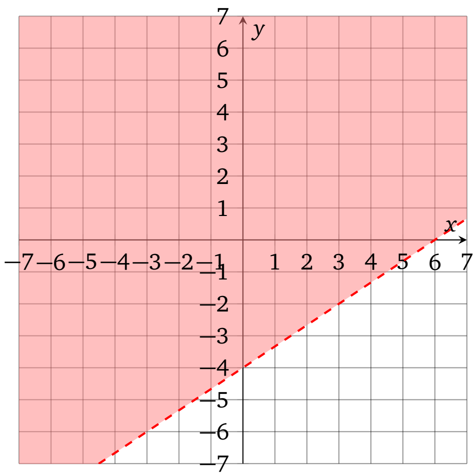

x
y
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
7 6 5 4 3 2 1 −1 −2 −3 −4 −5 −6 −7

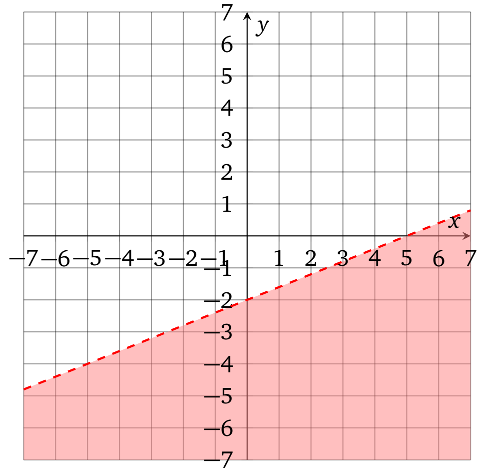
19.


x
y
−4 −3 −2 −1 1 2 3 4
4 3 2 1 −1 −2 −3 −4

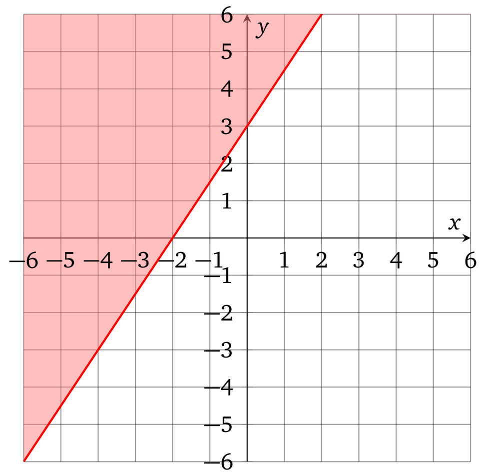
21.


x
y
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
7 6 5 4 3 2 1 −1 −2 −3 −4 −5 −6 −7

23.


x
y
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
7 6 5 4 3 2 1 −1 −2 −3 −4 −5 −6 −7

25.


x
y
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

Practice Test 4

1. Solve the system by graphing:

$$\begin{cases} 3y - x &= 3 \\ 2x - 4 &= y \end{cases}$$

2. Solve the system:

$$\begin{cases} 2x + y &= 12 \\ y - 3x &= 2 \end{cases}$$

3. Solve the system:

$$\begin{cases} -2x + 3y &= -1 \\ 2x + 5y &= 25 \end{cases}$$

4. Solve the system:

$$\begin{cases} y &= -3x - 4 \\ 0 &= 6x + 2y \end{cases}$$

5. A party with 3 adults and 2 children pays 45 dollars for their tickets at Acme Theater. Another party, with 4 adults and 4 children, pays 68 dollars. Find the price of one adult ticket and one the price of one child ticket.

6. A total of \$2400 is invested into simple interest accounts, part of it at 5% and the rest at 10%. The total interest after one year is \$175. How much was invested at each rate?

7. A pastry delivery is a mix of donuts and scones, 40 items in total. Each donut costs \$1.50, while each scone costs \$2.50. Find how many of each type of pastry there are if the total price of the order is \$76.

8. How many liters of 20% saline solution should be added to 40 liters of a 50% saline solution to make a 30% solution? How many liters of 30% solution will result?

9. Graph the solution set for inequality:

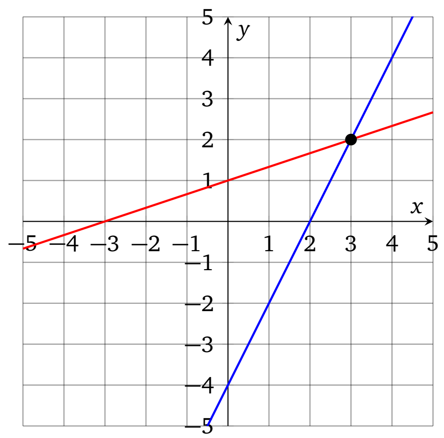
$$y < 5x - 2$$

10. Graph the solution set for inequality:

$$2y + 6 \geq -2(x + y - 3)$$

Practice Test 4 Answers.

1. $(3,2)$



2. $(2,8)$

3. $(5,3)$

4. no solutions

5.
a and c are prices of one adult and one child ticket respectively, in dollars

$$\begin{cases} 3x+2y=45 \\ 4x+4y=68 \end{cases}$$

solution: $a=11, \quad c=6$

6.
x and y are amounts invested at 5% and 10% respectively, in dollars

$$\begin{cases} x+y=2400 \\ 0.05x+0.1y=175 \end{cases}$$

solution: $x=1300, \quad y=1100$

7.
d and s are quantities of donuts and scones respectively

$$\begin{cases} d+s=40 \\ 1.5d+2.5s=76 \end{cases}$$

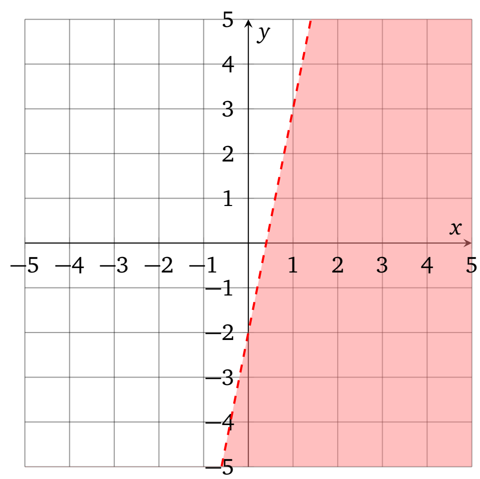
solution: $d=24, \quad s=16$

8.
x and y are volumes of 20% and 30% solutions respectively, in liters

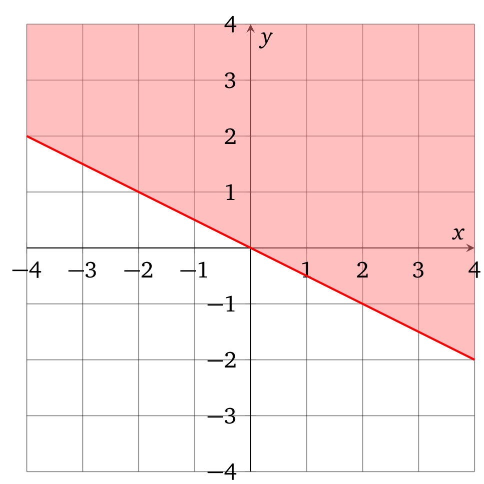
$$\begin{cases} x+40=y \\ 0.2x+0.5\cdot 40=0.3\cdot y \end{cases}$$

solution: $x=80, \quad y=120$

9.



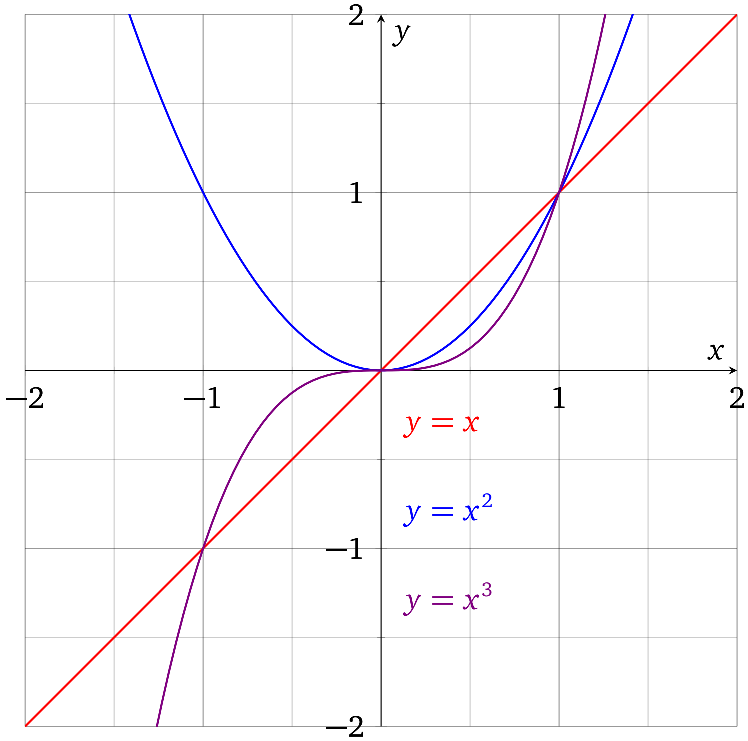
10.



CHAPTER 5

Polynomial Expressions

1. Integer Exponent



1.1. Definition.

DEFINITION 1.1.1 (Integer Exponent). Recall that given a real number a and a positive integer n, notation b^n means $\underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$. If $n = 0$ and b is any real number, then

$$b^n = b^0 = 1$$

If the exponent is negative and the base $b \neq 0$, then the reciprocal of the base is raised to the corresponding power:

$$b^{-n} = \left(\frac{1}{b}\right)^n = \frac{1}{b^n}$$

In particular, b^{-1} becomes another way to express the reciprocal of b.

BASIC EXAMPLE 1.1.1. Some trivial uses of the definition:

$$\begin{aligned}
x^1 &= x && \text{this is true for any real number } x \\
2^{-3} &= \left(\frac{1}{2}\right)^3 = \frac{1}{8} && \text{reciprocal of 2 cubed} \\
\left(-\frac{3}{7}\right)^{-1} &= -\frac{7}{3} && \text{exponent } -1 \text{ yields the reciprocal of the base} \\
(x^4 + 5x^2 + 13)^0 &= 1 && \text{exponent zero always results in 1 for any base} \\
\left(\frac{7}{5}\right)^{-2} &= \left(\frac{5}{7}\right)^2 = \frac{25}{49} && \text{take the reciprocal of the base, then square it}
\end{aligned}$$

THEOREM 1.1.1 (Exponent with Base ±1). The exponent with base 1 is always 1. If n is an integer, then

$$1^n = 1$$

It is also true that $(-1)^n = 1$ for all even n and $(-1)^n = -1$ for all odd n, even if the exponent n is negative.

When we simplify a product or a quotient of exponential expressions, we will attempt to bring it into the form with a single numerical coefficient, with each base appearing only once, and with all the exponents positive. We will also defer our exploration of the negative case until the end of this chapter, since polynomial expressions only make use of the non-negative integer exponent. It is of note, though, that the theorems which follow work for the negative exponent just as well.

1.2. Products and Quotients of Exponents.

THEOREM 1.2.1 (Product Rule for Exponents). A product of exponential expressions with the same base can be written as a single base with the sum of exponents. If m, n are integers and b is a real number, then

$$b^m b^n = b^{m+n}$$

BASIC EXAMPLE 1.2.1. It is easy enough to see why the product rule works if we follow the definition to simplify a product with the same base raised to various powers.

$$\begin{aligned}
b^3 b^2 &= (bbb)(bb) \\
&= bbbbb \\
&= b^5
\end{aligned}$$

EXAMPLE 1.2.1. Simplify the expression:

$$5^4 \cdot 5^3 \cdot 5$$

SOLUTION:

$$\begin{aligned} 5^4 \cdot 5^3 \cdot 5 &= 5^4 \cdot 5^3 \cdot 5^1 \\ &= 5^{4+3+1} \\ &= 5^8 \\ &= 390625 \end{aligned}$$

ANSWER: 390625

EXAMPLE 1.2.2. Simplify the expression:

$$(1+x)^5(1+x)^7$$

SOLUTION: Here the common base is $(1+x)$.

$$\begin{aligned} (1+x)^5(1+x)^7 &= (1+x)^{5+7} \\ &= (1+x)^{12} \end{aligned}$$

Recall that exponent does not distribute over addition, so we cannot simplify this answer any more.

ANSWER: $(1+x)^{12}$

EXAMPLE 1.2.3. Simplify the expression:

$$(a^2b^6)(ab^3)$$

SOLUTION: Factors of a product can be multiplied in any order, so we bring similar bases next to each other, then apply the product rule:

$$\begin{aligned} (a^2b^6)(ab^3) &= a^2a^1b^6b^3 \\ &= a^{2+1}b^{6+3} \\ &= a^3b^9 \end{aligned}$$

ANSWER: a^3b^9

EXAMPLE 1.2.4. Simplify the expression:

$$(-4x^3)(-3x^{17})$$

SOLUTION:

$$\begin{aligned}(-4x^3)(-3x^{17}) &= (-4)(-3)x^3x^{17}\\ &= 12x^{3+17}\\ &= 12x^{20}\end{aligned}$$

ANSWER: $12x^{20}$

THEOREM 1.2.2 (Quotient Rule for Exponents). A quotient of exponential expressions with the same base can be written as a single base with the difference of exponents. If m and n are integers and $b \neq 0$ is a real number, then

$$\frac{b^m}{b^n} = b^{m-n}$$

BASIC EXAMPLE 1.2.2. It is easy to see why the quotient rule works if we follow the definition.

$$\begin{aligned}\frac{b^6}{b^4} &= \frac{bbbbbb}{bbbb} && \text{four pairs of } b \text{ factors cancel}\\ &= bb\\ &= b^2\end{aligned}$$

BASIC EXAMPLE 1.2.3. The quotient rule is actually a direct consequence of definition of the negative exponent and the product rule. Recall that the fraction is in fact a product of the numerator and the reciprocal of the denominator:

$$\begin{aligned}\frac{b^m}{b^n} &= (b^m)\frac{1}{b^n} && \text{division is multiplication by reciprocal}\\ &= b^m b^{-n} && \text{by definition of negative exponent}\\ &= b^{m+(-n)} && \text{product rule}\\ &= b^{m-n}\end{aligned}$$

EXAMPLE 1.2.5. Simplify the expression:

$$\frac{5^{13}}{5^{10}}$$

SOLUTION:

$$\begin{aligned}\frac{5^{13}}{5^{10}} &= 5^{13-10} \\ &= 5^3 \\ &= 125\end{aligned}$$

ANSWER: 125

EXAMPLE 1.2.6. Simplify the expression:

$$\frac{5x^6y^8}{15x^2y}$$

SOLUTION:

$$\begin{aligned}\frac{5x^6y^8}{15x^2y} &= \frac{1}{3}\cdot\frac{x^6y^8}{x^2y^1} && \text{lowest terms} \\ &= \frac{1}{3}x^{6-2}y^{8-1} \\ &= \frac{1}{3}x^4y^7\end{aligned}$$

ANSWER: $\frac{1}{3}x^4y^7$

1.3. Exponents of Sums. It is a common mistake is to assume that exponent distributes over addition as well as over multiplication, but this is not the case. Students faced with an expression such as

$$(x+2)^2$$

are often tempted to rewrite it as x^2+2^2 or x^2+4, but this is not true in general, as anyone can check by comparing the values of these expressions for a specific x. For example, if $x=1$ then

$$\begin{aligned}(x+2)^2 &= (1+2)^2=3^2=9\\ x^2+4 &= 1^2+4=5\end{aligned}$$

So these expressions are not equivalent. Instead, when faced with exponents of sums, we will replace exponents by multiplication and then apply the distributivity of multiplication over addition:

$$\begin{aligned}(x+2)^2 &= (x+2)(x+2)\\ &= x\cdot(x+2)+2\cdot(x+2)\\ &= x\cdot x+x\cdot 2+2\cdot x+2\cdot 2\\ &= x^2+2x+2x+4\\ &= x^2+4x+4\end{aligned}$$

For now, though, we will leave answers like $(x+2)^2$ in exponential form.

Homework 5.1.

Simplify the given expression.

1. $(-1)^{1982}$

2. $(-1)^{1001}$

3. 0^0

4. $(1+x^2)^0$

5. $2^3 \cdot 2^4$

6. $3^3 \cdot 3^3$

7. $(-4)(-4)^2(-4)^3$

8. $(-5)^3(-5)^2$

9. $X^1X^{20}X^{300}$

10. $Y^{20}Y^0Y^{22}$

11. $(x-y)^6(x-y)^7$

12. $(a+2b)^{19}(a+2b)$

13. $(x^2y^7)(xy^2)$

14. $(n^3k^{10})(n^5k^4)$

15. $(-x^4y)(-6x^2y^{11})$

16. $(-5a^3b^0)(-20a^5b^7)$

17. $\dfrac{11^{46}}{11^{44}}$

18. $\dfrac{7^{108}}{7^{105}}$

19. $\dfrac{(-6)^{16}(-6)^4}{(-6)^{15}}$

20. $\dfrac{(-0.9)^{13}}{(-0.9)^2(-0.9)^8}$

21. $\dfrac{60x^8y^{12}}{-15x^5y^{11}}$

22. $\dfrac{8a^6y^5}{24x^5y^3}$

Homework 5.1 Answers.

1. 1

3. 1

5. 128

7. 4096

9. X^{321}

11. $(x-y)^{13}$

13. x^3y^9

15. $6x^6y^{12}$

17. 121

19. -7776

21. $-4x^3y$

2. Properties of Exponent

2.1. Exponent of Exponent.

THEOREM 2.1.1 (Power Rule for Exponents). When working with an exponent of an exponential expression, we can rewrite it as the same base with the product of the exponents. If m and n are integers and b is a real number, then

$$(b^m)^n = b^{m\cdot n}$$

BASIC EXAMPLE 2.1.1. It is easy to see why the power rule works if we follow the definition.

$$\begin{aligned} (b^2)^3 &= (bb)^3 \\ &= (bb)(bb)(bb) \\ &= bbbbbb \\ &= b^6 \end{aligned}$$

EXAMPLE 2.1.1. Simplify the expression: $(2^3)^4$

SOLUTION:

$$\begin{aligned} (2^3)^4 &= 2^{3\cdot 4} \\ &= 2^{12} \\ &= 4096 \end{aligned}$$

ANSWER: 4096

EXAMPLE 2.1.2. Simplify the expression:

$$((x^5)^3)^7$$

SOLUTION:

$$\begin{aligned} ((x^5)^3)^7 &= (x^{5\cdot 3})^7 \\ &= (x^{15})^7 \\ &= x^{15\cdot 7} \\ &= x^{105} \end{aligned}$$

ANSWER: x^{105}

2.2. Distributivity.

THEOREM 2.2.1 (Distributivity of Exponent over Multiplication). Integer exponent distributes over multiplication. If n is an integer and a, b are real numbers, then

$$(ab)^n = a^n b^n$$

This rule extends naturally to products with more than 2 factors:

$$(xyz)^n = x^n y^n z^n$$

BASIC EXAMPLE 2.2.1. It is easy to see why distributivity works if we follow the definition.

$$\begin{aligned} (ab)^4 &= (ab)(ab)(ab)(ab) \\ &= (aaaa)(bbbb) && \text{changed the order of multiplication} \\ &= a^4 b^4 \end{aligned}$$

EXAMPLE 2.2.1. Simplify the expression:

$$(-3x)^4$$

SOLUTION: The base of this exponential expression is a product with two factors: -3 and x, so we apply the distributive property:

$$\begin{aligned} (-3x)^4 &= (-3)^4 x^4 \\ &= 81x^4 \end{aligned}$$

ANSWER: $81x^4$

EXAMPLE 2.2.2. Simplify the expression:

$$(-10abc)^3$$

SOLUTION:

$$\begin{aligned} (-10abc)^3 &= (-10)^3 a^3 b^3 c^3 \\ &= -1000a^3 b^3 c^3 \end{aligned}$$

ANSWER: $-1000a^3 b^3 c^3$

EXAMPLE 2.2.3. Simplify the expression:

$$(x^3y^4)^5$$

SOLUTION: The base of the exponent has two factors: x^3 and y^4, so we distribute the exponent first, and then apply the rule for exponentiating the exponent:

$$\begin{aligned}(x^3y^4)^5 &= (x^3)^5(y^4)^5\\ &= x^{3\cdot 5}y^{4\cdot 5}\\ &= x^{15}y^{20}\end{aligned}$$

ANSWER: $x^{15}y^{20}$

EXAMPLE 2.2.4. Simplify the expression:

$$(-2ab^2)^3(a^3b)^5$$

SOLUTION: Distribute exponents over the products first, and simplify exponents of exponents:

$$\begin{aligned}(-2ab^2)^3(a^3b)^5 &= (-2)^3a^3(b^2)^3\cdot(a^3)^5b^5\\ &= -8a^3b^{2\cdot 3}\cdot a^{3\cdot 5}b^5\\ &= -8a^3b^6\cdot a^{15}b^5\end{aligned}$$

The factors of this product can be multiplied in any order, so we bring similar bases next to each other, and then apply the product rule:

$$\begin{aligned}-8a^3b^6\cdot a^{15}b^5 &= -8\cdot a^3a^{15}\cdot b^6b^5\\ &= -8\cdot a^{3+15}\cdot b^{6+5}\\ &= -8a^{18}b^{11}\end{aligned}$$

ANSWER: $-8a^{18}b^{11}$

EXAMPLE 2.2.5. Simplify the expression:

$$\frac{(4xy)^{17}}{(4xy)^{14}}$$

SOLUTION: Here the common base is $(4xy)$, so we can apply the quotient rule, and then distribute the exponent over the product. Alternatively, we could distribute first, and cancel common factors later.

$$\begin{aligned} \frac{(4xy)^{17}}{(4xy)^{14}} &= (4xy)^{17-14} \\ &= (4xy)^3 \\ &= 4^3x^3y^3 && \text{distribute exponent over the product} \\ &= 64x^3y^3 \end{aligned}$$

ANSWER: $64x^3y^3$

THEOREM 2.2.2 (Distributivity of Exponent in Fractions). Integer exponent distributes over division as well as over multiplication. If n is an integer, a is real, and b is a non-zero real, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

This rule extends naturally to fractions with more than 2 factors:

$$\left(\frac{2x}{abc}\right)^n = \frac{2^nx^n}{a^nb^nc^n}$$

BASIC EXAMPLE 2.2.2. It is easy to see why distributivity works if we follow the definition.

$$\begin{aligned} \left(\frac{a}{b}\right)^3 &= \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) \\ &= \frac{aaa}{bbb} \\ &= \frac{a^3}{b^3} \end{aligned}$$

EXAMPLE 2.2.6. Simplify the expression:

$$\left(\frac{7}{c}\right)^2$$

SOLUTION:

$$\begin{aligned}\left(\frac{7}{c}\right)^2 &= \frac{7^2}{c^2} \\ &= \frac{49}{c^2}\end{aligned}$$

ANSWER: $\frac{49}{c^2}$

EXAMPLE 2.2.7. Simplify the expression:

$$\left(\frac{-3d}{x^4}\right)^3$$

SOLUTION: The base of this exponential expression is a product of three factors: -3, d, and the reciprocal of x^4, so we can apply the distributive property:

$$\begin{aligned}\left(\frac{-3d}{x^4}\right)^3 &= \frac{(-3)^3 d^3}{(x^4)^3} \\ &= \frac{-27d^3}{x^{4\cdot 3}} && \text{power rule} \\ &= \frac{-27d^3}{x^{12}}\end{aligned}$$

ANSWER: $\frac{-27d^3}{x^{12}}$

EXAMPLE 2.2.8. Simplify the expression:

$$\frac{(-2bc^5)^3}{6b^2c^2}$$

SOLUTION: The base of the exponential expression in the numerator is a product of 3 factors: -2, b, and c^5. We have to distribute the exponent over this product before we can cancel common factors in the fraction:

$$\begin{aligned}
\frac{(-2bc^5)^3}{6b^2c^2} &= \frac{(-2)^3b^3(c^5)^3}{6b^2c^2} && \text{distributed exponent over the product} \\
&= \frac{-8b^3c^{5\cdot 3}}{6b^2c^2} && \text{power rule} \\
&= -\frac{4}{3}\frac{b^3c^{15}}{b^2c^2} && \text{lowest terms} \\
&= -\frac{4}{3}b^{3-2}c^{15-2} && \text{quotient rule} \\
&= -\frac{4}{3}bc^{13}
\end{aligned}$$

ANSWER: $-\frac{4}{3}bc^{13}$

Homework 5.2.

Simplify the given expression.

1. $(g^5)^8$

2. $(h^9)^{10}$

3. $(-2kx)^5$

4. $(cat)^7$

5. $3^0 + 5x^0$

6. $(2c^0)^4$

7. $(xy)^7(yz)^3$

8. $(a^2b)^4(a^5b^6)$

9. $(3m^3c^3)(-3m^{10}c^6)^2$

10. $(a^7b^4)^2(ab^2c)^9$

11. $\dfrac{(g^7h^2)^2}{(g^2h)^3}$

12. $\dfrac{(m^3n^2)^6}{(mn^3)^4}$

13. $(3^2)^3$

14. $((2^2)^2)^2$

15. $3^{(2^3)}$

16. $\mathrm{x}^{(3^4)}$

17. $\left((x^5y)(xy)^2\right)^3$

18. $\left((xy)^5(x^3y)^3\right)^2$

19. $\left(\dfrac{x^3}{-2xy^5}\right)^4$

20. $\left(\dfrac{a^5}{-3a^3b}\right)^3$

Homework 5.2 Answers.

1. g^{40}

3. $-32k^5x^5$

5. 6

7. $x^7y^{10}z^3$

9. $27m^{23}c^{15}$

11. g^8h

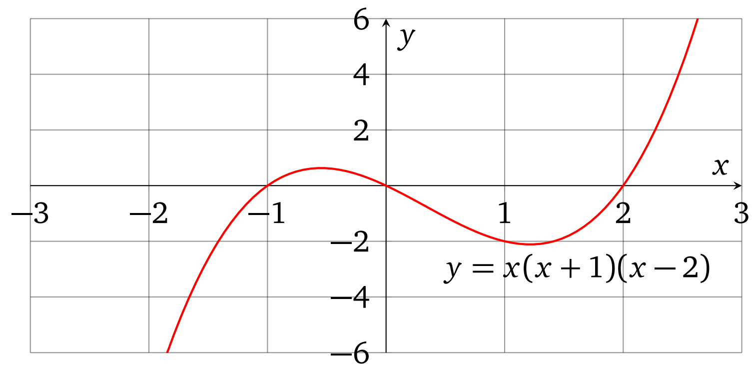
13. 729

15. 6561

17. $x^{21}y^9$

19. $\dfrac{x^8}{16y^{20}}$

3. Introduction to Polynomials



3.1. Definitions.

DEFINITION 3.1.1. A *monomial expression*, or just *monomial*, is an algebraic expression which is a product of numerical constants and/or variables, with each variable optionally carrying a positive integer exponent.

A *simplified form of a monomial expression* has a single numerical factor in the very front, called *coefficient*, and lists each variable base at most once.

BASIC EXAMPLE 3.1.1. This is a monomial with variables x and y and constants -3 and 2:

$$-3 \cdot x^2 \cdot y^5 \cdot x \cdot 2$$

To simplify, rearrange the factors and use the properties of exponent to combine the bases:

$$\begin{aligned} -3 \cdot x^2 \cdot y^5 \cdot x \cdot 2 &= (-3)(2) \cdot x^2 x^1 \cdot y^5 \\ &= -6 \cdot x^{2+1} \cdot y^5 \\ &= -6x^3y^5 \end{aligned}$$

In simplified form, this monomial has 3 factors and coefficient -6.

These are not monomials:

$x + y$, seen as a product, has a single factor $(x + y)$ which is neither a numerical constant nor a variable. In general, no sum with more than one term is a monomial.

$2/x$ has the reciprocal of x as a factor, which is the same as x^{-1}, but reciprocals of variables and negative exponents are not allowed.

$(3x)^2$ is not a monomial expression, since its only factor is neither a variable nor a numerical constant. It is though equivalent to $9x^2$, which is a monomial with two factors and coefficient 9.

DEFINITION 3.1.2. The *degree of a monomial* is the sum of all the exponents of the variables. Variables without visible exponents are considered to have exponent 1. Non-zero monomials with no variables have degree 0. The degree of the *zero monomial* is undefined. Descriptive names exist for monomials of low degrees, and learning the names of degrees 0 through 2 is particularly crucial for working with this text.

degree	**type of a monomial**
0	*constant*
1	*linear*
2	*quadratic*
3	cubic
4	quartic
5	quintic

BASIC EXAMPLE 3.1.2. Here are some monomials and their degrees.

$7xy$, which is equivalent to $7x^1y^1$, has degree $1+1=2$, so it is a quadratic monomial with coefficient 7.

-17 has degree 0, so it is a constant monomial with coefficient -17.

xy^2 has degree $1+2=3$, so it is a cubic monomial with coefficient 1.

$-xy^2z^5$ has degree $1+2+5=8$ and coefficient -1.

0 is the zero monomial. Its degree is traditionally left undefined in order to satisfy the following rule: the degree of a product of monomials is the sum of individual degrees. This rule works for any two non-zero monomials, but if we let the degree of the zero monomial to be 0, then any product involving it would also have degree 0, which is against the rule.

DEFINITION 3.1.3. A *polynomial expression*, or just *polynomial*, is a sum of monomial expressions. A *polynomial in one variable*, or *univariate polynomial*, uses at most one variable throughout the expression. A *polynomial in two variables*, or *bivariate polynomial*, uses at most two.

DEFINITION 3.1.4 (Standard Form for Polynomials). When writing down a polynomial in one variable x, it is traditional to order monomials in the descending order of degree, starting with the highest one, so the standard way of writing a polynomial like

$$x-3+2x^2-5x^7 \qquad \text{is} \qquad -5x^7+2x^2+x-3$$

The coefficient of the highest degree term is called the *leading coefficient* of the polynomial.

There are no firm conventions for writing down a polynomial in two or more variables, or for defining the leading coefficient in that case.

DEFINITION 3.1.5 (Polynomial Names). The following names exist for polynomials with a low number of terms:

number of terms	**type of a polynomial**
1	*monomial*
2	*binomial*
3	*trinomial*

DEFINITION 3.1.6. The *degree of a polynomial* is the highest of all the monomial degrees. The degree of the zero polynomial is undefined. The descriptive names for low degrees are the same as for monomials.

BASIC EXAMPLE 3.1.3. Here are some polynomials with their properties described:

polynomial	**number of terms**	**degree**	**description**
20	1	0	constant monomial
$x-3$	2	1	linear binomial
x^3+xyz	2	3	cubic binomial
x^2-x-1	3	2	quadratic trinomial
$-x^2y^2+xy^6+x^5$	3	7	7th degree trinomial

BASIC EXAMPLE 3.1.4. These are not polynomials:

$x-\frac{\mathbf{1}}{\mathbf{y}}$ has a term which is not a monomial.

$4(x+1)$ is not strictly speaking a polynomial expression, since it consists of a single term which is not a monomial. It is equivalent though to $4x+4$, which is a linear binomial, so later in the text we will refer to it as a polynomial in a fully factored form.

3.2. Combining Similar Monomial Terms.

DEFINITION 3.2.1. Monomial terms are *like* (used an an adjective) if their simplified forms consist of exactly the same variables raised to exactly the same powers. Sometimes like terms are also called *similar terms*.

EXAMPLE 3.2.1. Simplify the polynomial by combining like terms.

$$5x - 17x$$

SOLUTION: We combine like terms by applying the distributive property to their common factors, which is all the variables.

$$\begin{aligned} 5x - 17x &= (5-17)x \\ &= -12x \end{aligned}$$

ANSWER: $-12x$

EXAMPLE 3.2.2. Simplify the polynomial by combining like terms.

$$xy^2 + 13y^2x$$

SOLUTION: These terms don't *look* the same, but they are in fact similar, since we can rearrange the order of multiplication at will:

$$\begin{aligned} xy^2 + 13y^2x &= xy^2 + 13xy^2 \\ &= (1+13)xy^2 \\ &= 14xy^2 \end{aligned}$$

ANSWER: $14xy^2$

EXAMPLE 3.2.3. Simplify the polynomial by combining like terms.

$$3ab^3 - 5 + a^3b + 4 - 6ab^3$$

SOLUTION: Note that the terms $3ab^3$ and a^3b are not similar. Even though they have the same variables, the corresponding exponents are different. We can rearrange the terms of the sum so as to put like terms next to each other, and then apply the distributive property:

$$\begin{aligned} 3ab^3 - 5 + a^3b + 4 - 6ab^3 &= 3ab^3 - 6ab^3 - 5 + 4 + a^3b \\ &= (3-6)ab^3 - 1 + a^3b && -5+4=-1 \\ &= -3ab^3 - 1 + a^3b \end{aligned}$$

ANSWER: $-3ab^3 - 1 + a^3b$

EXAMPLE 3.2.4. Simplify the polynomial by combining like terms.

$$0.3x - 0.6 + 2.3x^2 - 1.7x - x^2 + 0.6$$

SOLUTION: Put like terms next to each other and distribute:

$$\begin{aligned} 0.3x - 0.6 + 2.3x^2 - 1.7x - x^2 + 0.6 &= 2.3x^2 - x^2 + 0.3x - 1.7x - 0.6 + 0.6 \\ &= (2.3 - 1)x^2 + (0.3 - 1.7)x + 0 \\ &= 1.3x^2 - 1.4x \end{aligned}$$

In answers, we will list the terms of polynomials in one variable in the descending order of degree.

ANSWER: $1.3x^2 - 1.4x$

3.3. Evaluating Polynomials in Applications.

EXAMPLE 3.3.1. Evaluate the expression $x^2 - 2x - 6$ for $x = 5$.

SOLUTION:

$$\begin{aligned} x^2 - 2x - 6 &= (5)^2 - 2 \cdot (5) - 6 \\ &= 25 - 10 - 6 \\ &= 9 \end{aligned}$$

ANSWER: 9

EXAMPLE 3.3.2. Evaluate the expression $-3x^2 + 2xy - 4y^2$ for $x = 3$ and $y = -2$.

SOLUTION:

$$\begin{aligned} -3x^2 + 2xy - 4y^2 &= -3 \cdot (3)^2 + 2(3)(-2) - 4(-2)^2 \\ &= (-3)(9) + (-12) - 4(4) \\ &= -27 - 12 - 16 \\ &= -55 \end{aligned}$$

ANSWER: -55

EXAMPLE 3.3.3. Professor calculates the class grade (out of 100 points) by computing the value of the following expression:

$$0.1H + 0.1Q + 0.6T + 0.2F$$

where H, Q, T, and F are the points accumulated for homework, quizzes, tests, and the final respectively. Find the class grade for a student with $H = 85$, $Q = 90$, $T = 76$, and $F = 68$.

SOLUTION:

$$\begin{aligned} 0.1H + 0.1Q + 0.6T + 0.2F &= 0.1(85) + 0.1(90) + 0.6(76) + 0.2(68) \\ &= 8.5 + 9 + 45.6 + 13.6 \\ &= 76.7 \end{aligned}$$

ANSWER: 76.7 points

EXAMPLE 3.3.4. The volume of a pyramid with a square base is given by the expression

$$\frac{1}{3}a^2h$$

where a is the side of the square and h is the height of the pyramid. Find the volume of the Great Pyramid of Giza, with dimensions $a = 230.4$ meters and $h = 146.5$ meters. (Actually, the pyramid's top deteriorated over time, so the actual height is a bit lower, but the historical height we are using here gives a better volume estimate.)

SOLUTION:

$$\begin{aligned} \frac{1}{3}a^2h &= \frac{1}{3}(230.4)^2 \cdot 146.5 \\ &= 2592276 \end{aligned}$$

ANSWER: 2592276 cubic meters

Homework 5.3.

Determine whether the given expression is a polynomial expression.

1. $7x-3$

2. $2x^5+9-7x^2$

3. $\dfrac{x^2+1}{x^2-5x+1}$

4. -11

5. x^5+x^{-5}

6. $\dfrac{1}{x}-\dfrac{2}{x^2}+x$

7. $(x+1)(x^2-10)$

8. $\dfrac{1}{3}x-\dfrac{2}{7}x^{10}$

9. $-\dfrac{2}{7}x^3+x^2+\dfrac{1}{7}$

10. $x^3-x(x+1)$

For each given polynomial, describe its degree and the number of terms in words. For example, $x+1$ is a *linear binomial*, while x^2+xy+y^2 is a *quadratic trinomial*. Use numbers for describing polynomials of high degree and/or with too many terms. For example, $x^{99}+x^3+x^2+x$ is a *polynomial of degree* 99 *with* 4 *terms*.

11. -5

12. $6x$

13. $5xy^2-3$

14. $xy+1-ab$

15. $a^2-\dfrac{1}{2}bc$

16. $-y^{14}$

17. $6x^3-3x^2+2x-1$

18. $3x^8+12x^3-8$

19. $-xy^2z^3$

20. $m^7+x^4y^4$

21. $a+b+c+d+3x$

22. $ab-cd-xy$

Combine the like terms. For polynomials in one variable, state answers in the standard form, with terms listed in the descending order of degree.

23. $6n^2-5+5n^2$

24. $4x+7x^2+3x$

25. $3x^4-2x+2x+x^4$

26. $9a^5+3a^2-2a^5-3a^2$

27. $10y^2+2y^3-3y^3-4y^2-6y^2-y^4$

28. $12b^6-b^3+8b^6+4b^3-b^7-3b^3$

29. $14x^2-5xy+xy^2-xy-7x^2+y^2x$

30. $-a^5b^2+4a^2b^5-2a^5b^2-6b^5a^2$

31. $9x^3+\dfrac{1}{2}x^2-\dfrac{2}{3}x+7x^3+\dfrac{5}{3}x+\dfrac{1}{4}x^2$

32. $\dfrac{1}{9}y^4-2y^3+\dfrac{12}{5}y-\dfrac{3}{5}y-\dfrac{1}{3}y^4+6y^3$

Evaluate each polynomial if $x = 3$, $y = -2$, and $z = 0.1$

33. $-3x + 9$

34. $12 - 10x$

35. $2x^2 - 3x + 6$

36. $-3x^2 - 4x + 5$

37. $2y^4 - \frac{1}{2}y^2$

38. $-y^3 + \frac{2}{5}y$

39. $x^2 - y^2z$

40. $x^4y^3z^2$

41. A chemist finds the amount (measured in liters) of acid in a mixture of x liters of solution A with y liters of solution B by computing the value of the expression

$$0.3x + 0.7y$$

Find the amount of acid in a mixture of $x = 1.2$ liters of solution A with $y = 0.85$ liters of solution B.

42. The distance in meters traveled by a projectile t seconds after the start of the timer can be found as the value of the following expression:

$$3.5 + 1.5t + \frac{5}{2}t^2$$

Find the distance traveled by the projectile 10 seconds after the start of the timer.

43. The amount of drug in the bloodstream, measured in mcg/mL, can be estimated by the value of the expression

$$-0.004t^4 + 0.004t^3 + 0.35t^2 + 0.6t$$

where t is the time in hours since the drug was administered. Find the amount of drug in the bloodstream 5 hours after the injection.

44. When the technology arm of Acme Co produces x computer monitors, the projected profit in dollars can be found using the expression

$$250x - 1.1x^2 - 4000$$

Find the profit corresponding to the production of 190 monitors.

Homework 5.3 Answers.

1. yes

3. no, because it is a fraction with a variable denominator

5. no, because it has negative exponents

7. no, even though it can be shown that this product of binomials is equivalent to a cubic polynomial

9. yes, because rational coefficients are OK

11. constant monomial

13. cubic binomial

15. quadratic binomial

17. cubic polynomial with 4 terms

19. 6th degree monomial

21. linear polynomial with 5 terms

23. $11n^2 - 5$

25. $4x^4$

27. $-y^4 - y^3$

29. $7x^2 - 6xy + 2xy^2$

31. $16x^3 + \frac{3}{4}x^2 + x$

33. 0

35. 15

37. 30

39. 8.6

41. 0.995 liters

43. 9.75 mcg/mL

4. Sums of Polynomials

We add polynomials by combining like terms.

EXAMPLE 4.0.1. Add polynomials by combining like terms:

$$(4x^2+6x-5)+(x^2-4x+5)$$

SOLUTION: Adding a sum amounts to adding each of its terms. To make like terms easier to see, we change the order of summation and put them next to each other:

$$\begin{aligned}(4x^2+6x-5)+(x^2-4x+5) &= 4x^2+6x-5+x^2-4x+5\\ &= 4x^2+x^2+6x-4x-5+5\\ &= (4+1)x^2+(6-4)x+0\\ &= 5x^2+2x\end{aligned}$$

ANSWER: $5x^2+2x$

BASIC EXAMPLE 4.0.1. Subtracting a polynomial amounts to adding its opposite. To rewrite the opposite of a polynomial as a polynomial expression, we remove the parentheses and invert the sign of each term:

$$-(x^7-x^4a^3-xa^6+7)=-x^7+x^4a^3+xa^6-7$$

EXAMPLE 4.0.2. Subtract polynomials by combining like terms:

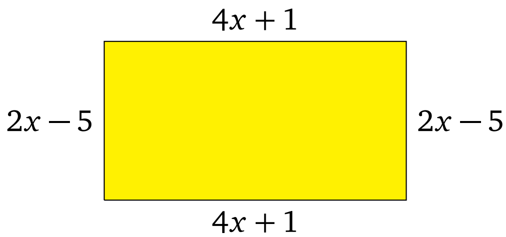
$$(5x^2-7x-6)-(2x^2+3x-4)$$

SOLUTION: The opposite of a sum is equal to the sum of opposites. This allows us to remove the parentheses on the second polynomial by taking the opposite of each term:

$$\begin{aligned}(5x^2-7x-6)-(2x^2+3x-4) &= 5x^2-7x-6-2x^2-3x+4\\ &= 5x^2-2x^2-7x-3x-6+4\\ &= (5-2)x^2+(-7-3)x-2\\ &= 3x^2-10x-2\end{aligned}$$

ANSWER: $3x^2-10x-2$

EXAMPLE 4.0.3. Find the perimeter of the shape by adding lengths of all sides:



SOLUTION: The terms of a sum can be added in any order, so we can put like terms next to each other and apply the distributive property.

$$\begin{aligned}(4x+1)+(2x-5)+(4x+1)+(2x-5) &= 4x+2x+4x+2x+1-5+1-5\\ &= (4+2+4+2)x-8\\ &= 12x-8\end{aligned}$$

ANSWER: $12x-8$

Homework 5.4.

Simplify the given expression and state the answer as a polynomial expression. If a polynomial is in one variable, list the terms of the answer in the decreasing order of degree.

1. $(3x+7)+(x+2)$

2. $(x+10)+(12x+1)$

3. $(2t+1)+(-8t+7)$

4. $(4t+2)+(-11t-3)$

5. $(7t^2-3t+9)+(2t^2+4t-6)$

6. $(8a^2+4a-7)+(6a^2-4a-1)$

7. $(3t^3+4t^2-1)+(-2t^2-4t+5)$

8. $(4.9x^3+3.2x^2-5.1x)+(2.1x^2-3.7x+4.5)$

9. $\left(\frac{4}{3}x^2+\frac{1}{2}x-3\right)+\left(\frac{1}{3}x^2-\frac{1}{8}x-\frac{1}{2}\right)$

10.

$\left(\frac{3}{5}y^4+\frac{1}{2}y^3-\frac{2}{3}y+3\right)+\left(\frac{2}{5}y^4-\frac{1}{4}y^3-\frac{3}{4}y^2\right)$

11. $(3x^2+6xy-xy^2-4)+(-xy-xy^2+4)$

12. $(7a^2b^2-6ab)+(a^4+a^2b^2+6ab)$

Rewrite the opposites of polynomials as polynomial expressions.

13. $-(-5x+7)$

14. $-(x^2+x)$

15. $-(3a^3-a^2+8b-7ab)$

16. $-(-12z^6-d^3-zd)$

Simplify the given expression and state the answer as a polynomial expression. If a polynomial is in one variable, list the terms of the answer in the decreasing order of degree.

17. $(3y+1)-(5y+8)$

18. $(7y-3)-(11y+17)$

19. $(4x^2+x-7)-(3-8x^2-4x^3)$

20. $(3a^3-2a+7)-(5a^2-2a^3+2a)$

21.

$\left(\frac{1}{5}y^3+2y^2-\frac{3}{10}\right)-\left(-\frac{2}{5}y^3+2y^2+\frac{7}{1000}\right)$

22. $\left(\frac{5}{8}y^3-\frac{1}{4}y-\frac{1}{3}\right)-\left(-\frac{1}{2}y^3+\frac{1}{4}y-\frac{1}{3}\right)$

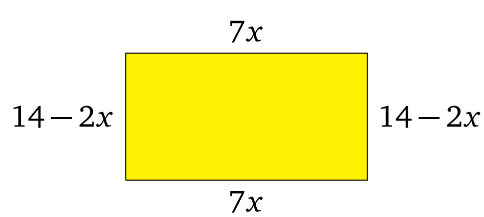
23. $(0.9x^3+0.2x-5)-(0.7x^4-0.3x-0.1)$

24.

$(0.07x^3-0.03x^2+0.01x)-(0.02x^3-0.04x^2-1)$

Find the perimeter of the shape by adding lengths of all sides.

25.



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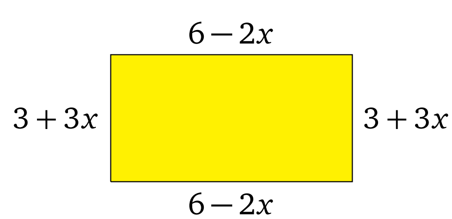

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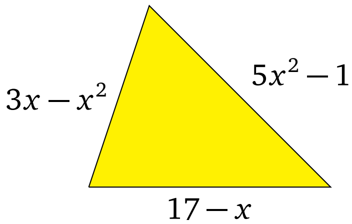

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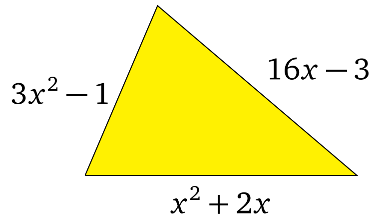


Homework 5.4 Answers.

1. $4x+9$

3. $-6t+8$

5. $9t^2+t+3$

7. $3t^3+2t^2-4t+4$

9. $\frac{5}{3}x^2+\frac{3}{8}x-\frac{7}{2}$

11. $3x^2+5xy-2xy^2$

13. $5x-7$

15. $-3a^3+a^2-8b+7ab$

17. $-2y-7$

19. $4x^3+12x^2+x-10$

21. $\frac{3}{5}y^3-\frac{307}{1000}$

23. $-0.7x^4+0.9x^3+0.5x-4.9$

25. $10x+28$

27. $4x^2+2x+16$

5. Products of Polynomials

5.1. Monomial Times Monomial. Since the order of multiplication does not affect the value of a product, we can multiply monomials by multiplying their coefficients and adding the variable powers using the product rule for exponent.

EXAMPLE 5.1.1. Simplify the expression:

$$(-5x^2y^6)(-2xy^3z^4)$$

SOLUTION:

$$\begin{aligned}(-5x^2y^6)(-2xy^3z^4) &= (-5)(-2)\cdot x^2x\cdot y^6y^3\cdot z^4 \\ &= 10\cdot x^{2+1}\cdot y^{6+3}\cdot z^4 && \text{product rule} \\ &= 10x^3y^9z^4\end{aligned}$$

ANSWER: $10x^3y^9z^4$

5.2. Monomial Times Polynomial.

EXAMPLE 5.2.1. Simplify the expression:

$$3x^2(5x^2-4x-1)$$

SOLUTION: This can be done by distributing, and then simplifying the resulting monomial products:

$$\begin{aligned}3x^2(5x^2-4x-1) &= (3x^2)(5x^2)-(3x^2)(4x)-(3x^2)(1) \\ &= 3\cdot 5\cdot x^2\cdot x^2-3\cdot 4\cdot x^2\cdot x-3\cdot 1\cdot x^2 \\ &= 15x^{2+2}-12x^{2+1}-3x^2 && \text{product rule} \\ &= 15x^4-12x^3-3x^2\end{aligned}$$

This is a univariate polynomial, so we state the answer in the standard form, by listing the terms in the descending order of their monomial degree:

ANSWER: $15x^4-12x^3-3x^2$

EXAMPLE 5.2.2. Simplify the expression:

$$-10xy^2(x^2 - 3xy + 2y^2)$$

SOLUTION: Notice that when we apply the distributive property, we have to compute the correct sign for each term:

$$\begin{aligned} -10xy^2(x^2 - 3xy + 2y^2) &= -(10xy^2)(x^2) + (10xy^2)(3xy) - (10xy^2)(2y^2) \\ &= -10 \cdot x^1x^2 \cdot y^2 + 30 \cdot x^1x^1 \cdot y^2y^1 - 20 \cdot x \cdot y^2y^2 \\ &= -10x^{1+2}y^2 + 30x^{1+1}y^{2+1} - 20xy^{2+2} \\ &= -10x^3y^2 + 30x^2y^3 - 20xy^4 \end{aligned}$$

This polynomial uses more than one variable, so there is no standard form, and the terms of the answer can be listed in any order:

ANSWER: $-10x^3y^2 + 30x^2y^3 - 20xy^4$

5.3. Polynomial Times Polynomial.

THEOREM 5.3.1. Multiplying two polynomials amounts to taking a sum of all monomials obtained by multiplying each term of the first polynomial by each term of the second polynomial. Since the product of any two monomials is a monomial, and a sum of monomials is a polynomial, it follows that the product of two polynomials is a polynomial.

EXAMPLE 5.3.1. Simplify the expression:

$$(3x - 2)(x + 5)$$

SOLUTION: We start by distributing the multiplication over the first sum:

$$(3x - 2)(x + 5) = 3x(x + 5) - 2(x + 5)$$

Then we distribute again as we multiply binomials by monomials:

$$\begin{aligned} 3x(x+5) - 2(x+5) &= (3x)(x) + (3x)(5) - (2)(x) - (2)(5) \\ &= 3x^2 + 15x - 2x - 10 && \text{simplified monomials} \\ &= 3x^2 + (15 - 2)x - 10 && \text{combined like terms} \\ &= 3x^2 + 13x - 10 \end{aligned}$$

ANSWER: $3x^2 + 13x - 10$

EXAMPLE 5.3.2. Simplify the expression:

$$(x+y-1)(2x+3y+5)$$

SOLUTION: This is a product of two trinomials, and we need to multiply each term of the first one by each term of the second one. This results in a sum of $3 \cdot 3 = 9$ monomials, which we simplify by combining the like terms.

$$\begin{aligned}(x+y-1)(2x+3y+5) &= x(2x+3y+5)+y(2x+3y+5)-1(2x+3y+5) \\ &= (2x^2+3xy+5x)+(2xy+3y^2+5y)+(-2x-3y-5) \\ &= 2x^2+3y^2+(3xy+2xy)+(5x-2x)+(5y-3y)-5 \\ &= 2x^2+3y^2+(3+2)xy+(5-2)x+(5-3)y-5 \\ &= 2x^2+3y^2+5xy+3x+2y-5\end{aligned}$$

This is a polynomial in 2 variables, so there is no standard form, and we can list the terms of the answer in any order.

ANSWER: $2x^2+3y^2+5xy+3x+2y-5$

There are many alternative ways to annotate the process of multiplying polynomials, like the table method, but of course they all produce the same result.

EXAMPLE 5.3.3. Simplify the expression by using a table:

$$(x+y-1)(2x+3y+5)$$

SOLUTION: We will construct a table where the top row will list the terms of the first polynomial, and the left column will list the terms of the second polynomial. We will then fill out the middle of the table by multiplying the monomials from the corresponding row and column. We will complete the task by adding all of the monomials from the middle of the table and combining like terms.

	$+x$	$+y$	-1
$+2x$	$+2x^2$	$+2xy$	$-2x$
$+3y$	$+3xy$	$+3y^2$	$-3y$
$+5$	$+5x$	$+5y$	-5

$$2x^2+2xy-2x+3xy+3y^2-3y+5x+5y-5=2x^2+3y^2+5xy+3x+2y-5$$

ANSWER: $2x^2+3y^2+5xy+3x+2y-5$

Homework 5.5.

Simplify the given expression and state the answer as a polynomial expression. If a polynomial is in one variable, list the terms of the answer in the decreasing order of degree.

1. $(4x^4)(8)$

2. $(-7)(6y^7)$

3. $(-x^5)(x^8)$

4. $(-y^3)(-2y^5)$

5. $\left(\frac{1}{5}t^2\right)\left(-\frac{2}{3}t^3\right)$

6. $\left(-\frac{3}{7}x^9\right)\left(-\frac{3}{2}x\right)$

7. $(-2x)(3x^2y)$

8. $(-1.5a^2b^2)(-2b)$

9. $(3x)(5x)(-2x^7)$

10. $\left(\frac{2}{3}x^2\right)(-6x)(-5x^4)$

11. $3x(x-y)$

12. $-6y(6-2xy)$

13. $3m(-m^2+2m-40)$

14. $12x(2x^3-x+3)$

15. $0.5b^3(0.2b^2-b+4)$

16. $0.1c^4(-4c^2+0.7c-10)$

17. $(x+3)(2x+3)$

18. $(3x+1)(x+5)$

19. $(x-0.6)(x-0.7)$

20. $(1.1x+10)(x-1.2)$

21. $(5x+6y^3)(x+y)$

22. $(x^5-2)(y^2-xy)$

23. $\left(\frac{1}{2}-x\right)\left(\frac{1}{3}x-4\right)$

24. $\left(2x-\frac{2}{3}\right)\left(\frac{3}{2}x+6\right)$

25. $(x+1)(x^2-x+1)$

26. $(x-2)(x^2+2x+4)$

27. $(x^2-x+3)(x+1)$

28. $(a^2+a-7)(a+2)$

29. $(5x^2-3x-1)(x^2+4)$

30. $(1+2a^2)(4a^2-5a-3)$

31. $(x+1)(x+2)(x+3)$

32. $5x^2(x+4)(x-4)$

33. $(a+b)^3$

34. $(x-2y)^3$

35. $(c^2-2c+3)(c^2+4c+1)$

36. $(-2x^2-x-1)(x^2-5x+1)$

Homework 5.5 Answers.

1. $32x^4$

3. $-x^{13}$

5. $-\frac{2}{15}t^5$

7. $-6x^3y$

9. $-30x^9$

11. $3x^2-3xy$

13. $-3m^3+6m^2-120m$

15. $0.1b^5-0.5b^4+2b^3$

17. $2x^2+9x+9$

19. $x^2-1.3x+0.42$

21. $5x^2+5xy+6xy^3+6y^4$

23. $-\frac{1}{3}x^2+\frac{25}{6}x-2$

25. x^3+1

27. x^3+2x+3

29. $5x^4-3x^3+19x^2-12x-4$

31. $x^3+6x^2+11x+6$

33. $a^3+3a^2b+3ab^2+b^3$

35. $c^4+2c^3-4c^2+10c+3$

6. Special Products

6.1. Difference of Squares.

THEOREM 6.1.1 (Difference of Squares Formula). For all real numbers A and B,

$$(A+B)(A-B) = A^2 - B^2$$

PROOF.

$$\begin{aligned} (A+B)(A-B) &= A(A-B)+B(A-B) \\ &= A^2 - AB + BA - B^2 && -AB \text{ and } BA \text{ are like terms} \\ &= A^2 - B^2 \end{aligned}$$

□

This is a nice computational shortcut which has many algebraic applications. In this text we will use it for factoring polynomial expressions and for simplifying radical expressions.

EXAMPLE 6.1.1. Simplify and state the result as a polynomial expression:

$$(2x+3)(2x-3)$$

SOLUTION: We will use the difference of squares formula $(A+B)(A-B) = A^2 - B^2$ with $A = 2x$ and $B = 3$.

$$\begin{aligned} (2x+3)(2x-3) &= (2x)^2 - (3)^2 \\ &= 2^2x^2 - 9 \\ &= 4x^2 - 9 \end{aligned}$$

ANSWER: $4x^2 - 9$

EXAMPLE 6.1.2. Simplify and state the result as a polynomial expression:

$$(x^3 - 5y^6)(x^3 + 5y^6)$$

SOLUTION: We will use the difference of squares formula $(A+B)(A-B) = A^2 - B^2$ with $A = x^3$ and $B = 5y^6$.

$$\begin{aligned}(x^3-5y^6)(x^3+5y^6) &= (x^3)^2-(5y^6)^2\\ &= x^{3\cdot 2}-5^2y^{6\cdot 2}\\ &= x^6-25y^{12}\end{aligned}$$

ANSWER: x^6-25y^{12}

EXAMPLE 6.1.3. Rewrite the difference of squares as a product of two binomials:

$$36-x^2$$

SOLUTION: $36=(6)^2$ and $x^2=(x)^2$, so we can use the difference of squares formula $(A+B)(A-B)=A^2-B^2$ with $A=6$ and $B=x$.

$$36-x^2=(6+x)(6-x)$$

ANSWER: $(6+x)(6-x)$

EXAMPLE 6.1.4. Rewrite the difference of squares as a product of two binomials:

$$4a^2-100b^6$$

SOLUTION: $4a^2=(2a)^2$ and $100b^6=(10b^3)^2$, so we can use the difference of squares formula $(A+B)(A-B)=A^2-B^2$ with $A=2a$ and $B=10b^3$.

$$\begin{aligned}4a^2-100b^6 &= (2a)^2-(10b^3)^2\\ &= (2a+10b^3)(2a-10b^3)\end{aligned}$$

ANSWER: $(2a+10b^3)(2a-10b^3)$

6.2. Square of a Binomial.

THEOREM 6.2.1 (Square of a Binomial). For all real numbers A and B, the square of a binomial sum can be computed using the following formula:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A^2+2AB+B^2\end{aligned}$$

The square of a binomial difference can be found using a similar formula:

$$\begin{aligned}(A-B)^2 &= (A-B)(A-B)\\ &= A^2-2AB+B^2\end{aligned}$$

PROOF.

$$\begin{aligned}(A+B)(A+B) &= A(A+B)+B(A+B)\\ &= A^2+AB+BA+B^2 && AB \text{ and } BA \text{ are like terms}\\ &= A^2+2AB+B^2\end{aligned}$$

$$\begin{aligned}(A-B)(A-B) &= A(A-B)-B(A-B)\\ &= A^2-AB-BA+B^2 && -AB \text{ and } -BA \text{ are like terms}\\ &= A^2-2AB+B^2\end{aligned}$$

□

EXAMPLE 6.2.1. Simplify and state the result as a polynomial expression:

$$(2a+4)^2$$

SOLUTION: We will use the square of a binomial formula

$$(A+B)^2 = A^2+2AB+B^2$$

with $A = 2a$ and $B = 4$:

$$\begin{aligned}(2a+4)^2 &= (2a)^2+2(2a)(4)+(4)^2\\ &= 2^2a^2+2\cdot 2\cdot 4\cdot a+16\\ &= 4a^2+16a+16\end{aligned}$$

This is a polynomial in one variable, so we leave the answer in the standard form, listing the terms in the descending order of their monomial degrees.

ANSWER: $4a^2+16a+16$

EXAMPLE 6.2.2. Simplify and state the result as a polynomial expression:

$$(7x-6y^5)^2$$

SOLUTION: We will use the square of a binomial formula

$$(A-B)^2 = A^2 - 2AB + B^2$$

with $A = 7x$ and $B = 6y^5$:

$$\begin{aligned}(7x-6y^5)^2 &= (7x)^2 - 2(7x)(6y^5) + (6y^5)^2\\ &= 7^2x^2 - 2\cdot 7\cdot 6\cdot xy^5 + 6^2y^{5\cdot 2}\\ &= 49x^2 - 84xy^5 + 36y^{10}\end{aligned}$$

ANSWER: $49x^2 - 84xy^5 + 36y^{10}$

There are other two sign patters for a binomial square, $(-A+B)^2$ and $(-A-B)^2$, and they can be reduced to the ones we already discussed.

EXAMPLE 6.2.3. Simplify and state the result as a polynomial expression: $(-x+10)^2$

SOLUTION:

$$\begin{aligned}(-x+10)^2 &= (10-x)^2 && \text{same as } (A-B)^2\\ &= (10)^2 - 2(10)(x) + (x)^2\\ &= 100 - 20x + x^2\end{aligned}$$

ANSWER: $100 - 20x + x^2$

EXAMPLE 6.2.4. Simplify and state the result as a polynomial expression: $(-5-xy)^2$

SOLUTION:

$$\begin{aligned}(-5-xy)^2 &= ((-5)+(-xy))^2 && \text{same as } (A+B)^2\\ &= (-5)^2 + 2(-5)(-xy) + (-xy)^2 && \text{all minuses cancel}\\ &= 25 + 10xy + x^2y^2\end{aligned}$$

ANSWER: $25 + 10xy + x^2y^2$

Homework 5.6.

Use appropriate special product formulas to simplify the given expression and state the answer as a polynomial expression. If a polynomial is in one variable, list the terms of the answer in the decreasing order of degree.

1. $(5a+3)^2$

2. $(1+4x)^2$

3. $(x+10)(x-10)$

4. $(9-a)(9+a)$

5. $(4x-2y)^2$

6. $(a^2-3)^2$

7. $(2b+5c)(2b-5c)$

8. $(p^3-7)(p^3+7)$

9. $(2nb-5b)^2$

10. $(5-6x^2y)^2$

11. $(2x+xy^3)^2$

12. $(y^4+x^2)^2$

13. $(0.5x-0.4)(0.5x+0.4)$

14. $(2.5x+3.1y)(2.5x-3.1y)$

15. $\left(3+\frac{1}{2}x^7\right)^2$

16. $\left(\frac{1}{3}+\frac{1}{5}x\right)^2$

17. $\left(\frac{1}{4}x^4-\frac{6}{5}x\right)^2$

18. $\left(\frac{2}{3}y-\frac{4}{3}y^3\right)^2$

19. $(-7-20x)^2$

20. $(-2x-0.4y)^2$

21. $\left(-\frac{12}{5}+\frac{15}{2}c\right)^2$

22. $\left(-\frac{5}{3}d^6+6d^3\right)^2$

23. $((2x+1)(2x-1))^2$

24. $((0.5y-4x)(0.5y+4x))^2$

Rewrite the difference of squares as a product of two binomials.

25. $36x^2-y^2$

26. $4a^2-9b^2$

27. c^4-d^{10}

28. m^6-16n^8

29. $25k^2s^2-0.01k^4s^2$

30. $49x^8y^2-\frac{1}{4}x^4y^4$

Homework 5.6 Answers.

1. $25a^2+30a+9$

3. x^2-100

5. $16x^2-16xy+4y^2$

7. $4b^2-25c^2$

9. $4n^2b^2-20nb^2+25b^2$

11. $4x^2+4x^2y^3+x^2y^6$

13. $0.25x^2-0.16$

15. $9+3x^7+\frac{1}{4}x^{14}$

17. $\frac{1}{16}x^8-\frac{3}{5}x^5+\frac{36}{25}x^2$

19. $49+280x+400x^2$

21. $\frac{225}{4}c^2-36c+\frac{144}{25}$

23. $16x^4-8x^2+1$

25. $(6x+y)(6x-y)$

27. $(c^2+d^5)(c^2-d^5)$

29. $(5ks+0.1k^2s)(5ks-0.1k^2s)$

7. Quotients of Polynomials

7.1. Monomial Divisor.

THEOREM 7.1.1. A division of a polynomial by a monomial can be rewritten as a sum of monomial quotients with a common denominator. For all monomials A and B and all non-zero monomials C,

$$\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C}$$

This result naturally generalizes to sums with more than two terms.

PROOF.

$$\begin{aligned} \frac{A+B}{C} &= (A+B)\cdot\frac{1}{C} \\ &= A\cdot\frac{1}{C} + B\cdot\frac{1}{C} \\ &= \frac{A}{C} + \frac{B}{C} \end{aligned}$$

□

EXAMPLE 7.1.1. Simplify and state the result as a polynomial expression:

$$\frac{x^4 - 2x^3 + 6x^2}{2x}$$

SOLUTION: We rewrite the division as a sum of monomial quotients and then simplify the resulting fractions by canceling common factors:

$$\begin{aligned} \frac{x^4 - 2x^3 + 6x^2}{2x} &= \frac{x^4}{2x} - \frac{2x^3}{2x} + \frac{6x^2}{2x} \\ &= \frac{1}{2}\cdot\frac{x^4}{x} - \frac{x^3}{x} + 3\cdot\frac{x^2}{x} && \text{lowest terms} \\ &= \frac{1}{2}x^3 - x^2 + 3x \end{aligned}$$

This is a polynomial in one variable, so we leave the answer in the standard form, listing the terms in the descending order of their monomial degrees.

ANSWER: $\frac{1}{2}x^3 - x^2 + 3x$

EXAMPLE 7.1.2. Simplify and state the result as a polynomial expression:

$$\frac{-10a^5b^6+6a^3b^4+5ab^2}{-5ab^2}$$

SOLUTION: We rewrite the division as a sum of monomial quotients and then simplify the resulting fractions by canceling common factors. Notice that we compute the correct signs for each term just as we write the sum of quotients:

$$\begin{aligned}\frac{-10a^5b^6+6a^3b^4+5ab^2}{-5ab^2} &= \frac{10a^5b^6}{5ab^2}-\frac{6a^3b^4}{5ab^2}-\frac{5ab^2}{5ab^2} && \text{cancel common factors}\\ &= 2a^4b^4-\frac{6}{5}a^2b^2-1 && \text{in each fraction}\end{aligned}$$

ANSWER: $2a^4b^4-\frac{6}{5}a^2b^2-1$

Unlike monomial products, not all monomial quotients are monomial. In general, a so-called rational expression results, and we dedicate a whole chapter to them in this text. For now we take a sneak peek at how this happens.

EXAMPLE 7.1.3. Simplify and state the result as a sum of simplified monomial quotients:

$$\frac{-p^5s-7p^2s^3+p}{-p^2s^2}$$

SOLUTION: We rewrite the division as a sum of monomial quotients and then cancel common factors in each term.

$$\begin{aligned}\frac{-p^5s-7p^2s^3+p}{-p^2s^2} &= \frac{p^5s}{p^2s^2}+\frac{7p^2s^3}{p^2s^2}-\frac{p}{p^2s^2}\\ &= \frac{p^3}{s}+7s-\frac{1}{ps^2}\end{aligned}$$

The terms p^3/s and $-1/(ps^2)$ contain variable reciprocals, so they are not monomial, but proper rational expressions in simplified form.

ANSWER: $\frac{p^3}{s}+7s-\frac{1}{ps^2}$

7.2. Polynomial Divisor. There exists a division algorithm for polynomials, which is useful for rewriting them as products of polynomial factors.

THEOREM 7.2.1. If A is a polynomial and B is a non-zero polynomial in one variable, and the degree of A is at least as big as the degree of B, then

$$\frac{A}{B} = Q + \frac{R}{B}$$

where Q and R are polynomials and the degree of R is less than the degree of B. If the polynomial remainder R happens to be zero, then $A/B = Q$ is a polynomial quotient of two polynomials.

The polynomial long division algorithm will be explained in examples. In each case, we will state the result in the form $Q + R/B$, just as in the theorem (7.2.1).

EXAMPLE 7.2.1. Simplify using polynomial long division:

$$\frac{x^2 + 5x + 6}{x + 3}$$

SOLUTION: We begin by setting up the long division. It resembles the integer long division, but instead of columns with digits it has columns of monomial terms. Starting from the right and going to the left, the powers of x are 0, 1, 2, and so on, as many as the dividend monomial has.

$$x+3\overline{\big)\; x^2 + 5x + 6}$$

Divide the leftmost monomial of the dividend by the leftmost monomial of the divisor: $x^2/x = x$. Write the result in the appropriate column of the answer space.

$$\begin{array}{r} x \phantom{{}+5x+6} \\ x+3\overline{\big)\; x^2 + 5x + 6} \end{array}$$

Take the opposite of the monomial you just found and multiply it by the divisor $x + 3$:

$$(-x)(x + 3) = -x^2 - 3x$$

Write the result below the dividend in appropriate columns.

$$\begin{array}{r} x \phantom{{}+5x+6} \\ x+3\overline{\big)\; x^2 + 5x + 6} \\ \underline{-x^2 - 3x} \phantom{{}+6} \end{array}$$

Add the monomials above the line. No need to write zero for $x^2 + (-x^2)$, since that place always cancels. Carry down the next monomial term from the dividend.

$$
\begin{array}{r}
x \\
x+3 \overline{)\; x^2 + 5x + 6} \\
\underline{-x^2 - 3x} \phantom{{}+6} \\
2x + 6
\end{array}
$$

Now we repeat the steps with $2x + 6$ as the new dividend. Divide the leftmost monomial of the dividend by the leftmost monomial of the divisor: $2x/x = 2$, and write the result in the answer space.

$$
\begin{array}{r}
x + 2 \\
x+3 \overline{)\; x^2 + 5x + 6} \\
\underline{-x^2 - 3x} \phantom{{}+6} \\
2x + 6
\end{array}
$$

Take the opposite of the monomial you just found and multiply it by the divisor $x + 3$:

$$(-2)(x + 3) = -2x - 6$$

Write the result below the dividend in appropriate columns.

$$
\begin{array}{r}
x + 2 \\
x+3 \overline{)\; x^2 + 5x + 6} \\
\underline{-x^2 - 3x} \phantom{{}+6} \\
2x + 6 \\
\underline{-2x - 6}
\end{array}
$$

Add the monomials above the line.

$$
\begin{array}{r}
x + 2 \\
x+3 \overline{)\; x^2 + 5x + 6} \\
\underline{-x^2 - 3x} \phantom{{}+6} \\
2x + 6 \\
\underline{-2x - 6} \\
0
\end{array}
$$

There's nothing more to carry down, so we have reached the answer. The quotient $x + 2$ is written in the answer space. The remainder 0 is at the very bottom. We state the answer as a quotient without a remainder.

ANSWER: $x + 2$

EXAMPLE 7.2.2. Simplify using polynomial long division:

$$\frac{x^3 - 2x^2 - 4}{x - 3}$$

SOLUTION: We begin by setting up the long division. Notice the empty space between the terms $-2x^2$ and -4 of the dividend. This column is reserved for monomials of degree 1. Some people prefer to fill the zero entries below the dividend with visible zeroes, but we are just leaving them blank.

$$x - 3\overline{)\; x^3 - 2x^2 \qquad - 4}$$

Divide the leftmost monomial of the dividend by the leftmost monomial of the divisor: $x^3/x = x^2$. Write the result in the appropriate column of the answer space.

$$\begin{array}{rl} & x^2 \\ x-3 \big) & \overline{x^3 - 2x^2 \qquad - 4} \end{array}$$

Take the opposite of the monomial you just found and multiply it by the divisor $x - 3$:

$$(-x^2)(x - 3) = -x^3 + 3x^2$$

Write the result below the dividend in appropriate columns.

$$\begin{array}{rl} & x^2 \\ x-3 \big) & \overline{x^3 - 2x^2 \qquad - 4} \\ & \underline{-x^3 + 3x^2} \end{array}$$

Add the monomials above the line. Carry down the next monomial term from the dividend. The next term to be carried down is $0x$, so we just carry down the empty space.

$$\begin{array}{rl} & x^2 \\ x-3 \big) & \overline{x^3 - 2x^2 \qquad - 4} \\ & \underline{-x^3 + 3x^2} \\ & \qquad x^2 \end{array}$$

Repeat the steps with $2x + 6$ as the new dividend. Divide the leftmost monomial of the dividend by the leftmost monomial of the divisor: $x^2/x = x$, and write the result in the appropriate column of the answer space.

$$
\begin{array}{r}
x^2 \quad + x \\
x-3 \overline{)\; x^3 - 2x^2 \qquad - 4} \\
\underline{- x^3 + 3x^2} \\
x^2
\end{array}
$$

Take the opposite of the monomial you just found and multiply it by the divisor $x-3$:

$$(-x)(x-3) = -x^2 + 3x$$

Write the result below the dividend in appropriate columns.

$$
\begin{array}{r}
x^2 \quad + x \\
x-3 \overline{)\; x^3 - 2x^2 \qquad - 4} \\
\underline{- x^3 + 3x^2} \\
x^2 \\
\underline{- x^2 + 3x}
\end{array}
$$

Add the monomials above the line. Carry down the next monomial term from the dividend.

$$
\begin{array}{r}
x^2 \quad + x \\
x-3 \overline{)\; x^3 - 2x^2 \qquad - 4} \\
\underline{- x^3 + 3x^2} \\
x^2 \\
\underline{- x^2 + 3x} \\
3x - 4
\end{array}
$$

Repeat the steps with $3x-4$ as the new dividend. Divide the leftmost monomial of the dividend by the leftmost monomial of the divisor: $3x/x = 3$, and write the result in the answer space.

$$
\begin{array}{r}
x^2 \quad + x + 3 \\
x-3 \overline{)\; x^3 - 2x^2 \qquad - 4} \\
\underline{- x^3 + 3x^2} \\
x^2 \\
\underline{- x^2 + 3x} \\
3x - 4
\end{array}
$$

Take the opposite of the monomial you just found and multiply it by the divisor $x-3$:

$$(-3)(x-3) = -3x + 9$$

Write the result below the dividend in appropriate columns.

$$
\begin{array}{rr}
 & x^2 \quad +x+3 \\ \cline{2-2}
x-3\big) & x^3-2x^2 \qquad -4 \\
 & \underline{-x^3+3x^2} \qquad\qquad \\
 & x^2 \qquad\qquad \\
 & \underline{-x^2+3x} \qquad \\
 & 3x-4 \\
 & \underline{-3x+9}
\end{array}
$$

Add the monomials above the line.

$$
\begin{array}{rr}
 & x^2 \quad +x+3 \\ \cline{2-2}
x-3\big) & x^3-2x^2 \qquad -4 \\
 & \underline{-x^3+3x^2} \qquad\qquad \\
 & x^2 \qquad\qquad \\
 & \underline{-x^2+3x} \qquad \\
 & 3x-4 \\
 & \underline{-3x+9} \\
 & 5
\end{array}
$$

There's nothing more to carry down, so we have reached the answer. The quotient x^2+x+3 is written in the answer space. The remainder 5 is at the very bottom. We state the answer in the form $Q+R/B$, where Q is the polynomial quotient, R is the polynomial remainder, and R is the original polynomial divisor.

ANSWER: $x^2+x+3+\dfrac{5}{x-3}$

Checking these answers is possible, but not easy. One way to do that is to multiply the answer by the divisor, simplify, and make sure we get the original dividend back:

$$
\begin{aligned}
\left(x^2+x+3+\frac{5}{x-3}\right)(x-3) &= x^2(x-3)+x(x-3)+3(x-3)+\frac{5(x-3)}{x-3} \\
&= x^3-3x^2+x^2-3x+3x-9+5 \\
&= x^3+(-3+1)x^2+(-3+3)x-4 \\
&= x^3-2x^2-4
\end{aligned}
$$

We will present a few more examples of polynomial long division without explaining the steps. The procedure is exactly the same in each case. Readers can do these on their own, and then compare the solutions in order to check the answer and to detect where the first mistake was made.

EXAMPLE 7.2.3. Simplify using polynomial long division: $\dfrac{6x^2-5x-27}{2x-5}$

SOLUTION:

$$\begin{array}{r|rrr}
 & 3x & +5 & \\
\hline
2x-5 & 6x^2 & -5x & -27 \\
 & -6x^2 & +15x & \\
\hline
 & & 10x & -27 \\
 & & -10x & +25 \\
\hline
 & & & -2
\end{array}$$

ANSWER: $3x+5+\dfrac{-2}{2x-5}$

Checking the answer:

$$\begin{aligned}
\left(3x+5+\frac{-2}{2x-5}\right)(2x-5) &= 3x(2x-5)+5(2x-5)+\frac{-2(2x-5)}{(2x-5)} \\
&= 6x^2-15x+10x-25+(-2) \\
&= 6x^2-5x-27
\end{aligned}$$

EXAMPLE 7.2.4. Simplify using polynomial long division: $\dfrac{x^4-2x^3+3x-4}{x^2-5}$

SOLUTION:

$$\begin{array}{r|rrrrr}
 & & & x^2 & -2x & +8 \\
\hline
x^2-5 & x^4 & -2x^3 & +3x^2 & & -4 \\
 & -x^4 & & +5x^2 & & \\
\hline
 & & -2x^3 & +8x^2 & & \\
 & & 2x^3 & & -10x & \\
\hline
 & & & 8x^2 & -10x & -4 \\
 & & & -8x^2 & & +40 \\
\hline
 & & & & -10x & +36
\end{array}$$

ANSWER: $x^2-2x+8+\dfrac{-10x+36}{x^2-5}$

Homework 5.7.

Simplify the given expression and state the result as a polynomial expression. If a polynomial is in one variable, list the terms of the answer in the decreasing order of degree.

1. $\dfrac{42x^6-24}{4}$

2. $\dfrac{8a^2+10a}{16}$

3. $\dfrac{x-2x^2+x^7}{x}$

4. $\dfrac{4y^{10}-2y^3+y^2}{y^2}$

5. $\dfrac{20t^3-25t^2+10t}{-5t}$

6. $\dfrac{14y^5+28y^4-70y^3}{-14y}$

7. $\dfrac{-12x^2y^2+4x^4y^2+16xy^3}{4xy}$

8. $\dfrac{9m^2n^2-3m^2n-6mn^2}{-3mn}$

9. $\dfrac{20ab^2c^3-14a^2b^2c^4}{10ab^2c^2}$

10. $\dfrac{17x^3yz^5+16x^2yz^6}{2x^2yz}$

Simplify the expression using polynomial long division. State the answer as a polynomial expression when the remainder is zero; otherwise use the form $Q+R/B$, where Q is the polynomial quotient, B is the polynomial divisor, and the degree of the polynomial R is lower than the degree of B.

11. $\dfrac{x^2+2x-15}{x+5}$

12. $\dfrac{x^2-8x+12}{x-6}$

13. $\dfrac{4x^2-6x+7}{-x-3}$

14. $\dfrac{3y^2+5y+4}{-y+5}$

15. $\dfrac{x^2-10x-20}{x-5}$

16. $\dfrac{t^2+8t-15}{t+6}$

17. $\dfrac{6y^2+17y+8}{2y+5}$

18. $\dfrac{10a^2+19a+10}{2a+3}$

19. $\dfrac{t^3+64}{t+4}$

20. $\dfrac{27-x^3}{3-x}$

21. $\dfrac{a^3-a^2+a-1}{a^2+1}$

22. $\dfrac{8t^3-22t^2-5t+12}{2t^2-7t+4}$

Homework 5.7 Answers.

1. $10.5x^6 - 6$

3. $x^6 - 2x + 1$

5. $-4t^2 + 5t - 2$

7. $-3xy + x^3y + 4y^2$

9. $2c - 1.4ac^2$

11. $x - 3$

13. $-4x + 18 + \frac{61}{-x-3}$

15. $x - 5 + \frac{-45}{x-5}$

17. $3y + 1 + \frac{3}{2y+5}$

19. $t^2 - 4t + 16$

21. $a - 1$

8. Negative Exponent

8.1. Negative Integer Exponent. Recall that by definition of negative integer exponent, if b is any non-zero real number and n is a positive integer, then

$$b^{-n} = \left(\frac{1}{b}\right)^n = \frac{1}{b^n}$$

Recall also that negative integer exponent obeys all of the exponential rules we've seen so far:

(1) The value of exponent with base 1 is 1
(2) The product rule and the quotient rule
(3) The power rule
(4) Distributivity over multiplication and division

In this section we will practice manipulating products and quotients involving negative exponent by rewriting them in the form where each variable base appears only once, and all exponents are positive.

BASIC EXAMPLE 8.1.1. We start with some basic examples. By definition, raising a base to a negative power amounts to raising the reciprocal of the base to the corresponding positive power:

$$4^{-2} = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

Next, observe what happens to the sign of the base. The reciprocal of the negative number is negative, so the sign of the exponent has no influence on the sign of the base. What does make a difference is the oddness of the exponent. An odd exponent of any sign will preserve the sign of the base, and an even exponent will always produce a non-negative result:

$$(-5)^{-3} = \left(-\frac{1}{5}\right)^3 = \left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) = -\frac{1}{125}$$

$$(-3)^{-4} = \left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \frac{1}{81}$$

Rewriting variable exponents is even easier, since we do not have to compute their values:

$$a^3b^{-2} = a^3 \cdot \left(\frac{1}{b}\right)^2 = a^3 \cdot \left(\frac{1}{b^2}\right) = \frac{a^3}{b^2}$$

In general, getting rid of the negative exponent within products and fractions is quite straightforward, thanks to the following fact.

THEOREM 8.1.1. Within a product or a fraction, a factor in the numerator with negative exponent can be replaced by the same factor in the denominator with positive exponent. For all real numbers a, all non-zero real numbers b and X, and all integers n,

$$\frac{aX^{-n}}{b} = \frac{a}{bX^n}$$

Conversely, a factor in the denominator with negative exponent can be replaced by the same factor in the numerator with positive exponent.

$$\frac{a}{bX^{-n}} = \frac{aX^n}{b}$$

PROOF.

First part:

$$\begin{aligned}\frac{aX^{-n}}{b} &= \frac{a}{b}\cdot(X^{-n}) \\ &= \frac{a}{b}\cdot\left(\frac{1}{X}\right)^n \\ &= \frac{a}{b}\cdot\left(\frac{1}{X^n}\right) \\ &= \frac{a}{bX^n}\end{aligned}$$

Second part:

$$\begin{aligned}\frac{a}{bX^{-n}} &= \frac{a}{b}\cdot\left(\frac{1}{X^{-n}}\right) \\ &= \frac{a}{b}\cdot\left(\frac{1}{X}\right)^{-n} \\ &= \frac{a}{b}\cdot(X^n) \\ &= \frac{aX^n}{b}\end{aligned}$$

□

EXAMPLE 8.1.1. Simplify and state the answer as an expression with positive exponents:

$$\frac{x^{-17}y^5}{x^{-13}y^{-6}}$$

SOLUTION: One way to deal with negative exponent factors is by replacing them with corresponding positive exponent factors, using the theorem 8.1.1:

$$\begin{aligned}\frac{x^{-17}y^5}{x^{-13}y^{-6}} &= \frac{x^{13}y^5y^6}{x^{17}} \\ &= \frac{y^{5+6}}{x^4} && \text{common factor } x^{13} \text{ canceled} \\ &= \frac{y^{11}}{x^4}\end{aligned}$$

Another way to deal with this is by applying the product and the quotient rules directly to the negative exponent. Here we are diving bases, so we will subtract the corresponding exponents:

$$\begin{aligned}\frac{x^{-17}y^5}{x^{-13}y^{-6}} &= x^{-17-(-13)}y^{5-(-6)} \\ &= x^{-4}y^{11} \\ &= \frac{y^{11}}{x^4} && \text{got rid of the negative exponent for the answer}\end{aligned}$$

Both ways are correct and about as efficient, and choosing one over the other, or a mix of the two, is more of a personal preference.

ANSWER: $\frac{y^{11}}{x^4}$

EXAMPLE 8.1.2. Simplify and state the answer as an expression with positive exponents:

$$(5x^7x^{-4})^{-2}$$

SOLUTION: We will simplify inside the parentheses first since we can. Alternatively, we could distribute the exponent over the product first.

$$\begin{aligned}(5x^7x^{-4})^{-2} &= (5x^{7+(-4)})^{-2} \\ &= (5x^3)^{-2}\end{aligned}$$

Now we distribute exponent over the simplified product:

$$
\begin{aligned}
&= 5^{-2} \cdot (x^3)^{-2} && \\
&= \frac{1}{5^2} \cdot x^{3\cdot(-2)} && \text{product rule} \\
&= \frac{1}{25} \cdot x^{-6} && \\
&= \frac{1}{25x^6} &&
\end{aligned}
$$

ANSWER: $\frac{1}{25x^6}$

EXAMPLE 8.1.3. Simplify and state the answer as an expression with positive exponents:

$$\frac{-10x^7y^{20}}{(5x^3y^{10})^2}$$

SOLUTION: We have to distribute the exponent over the product in the denominator before we can cancel anything:

$$
\begin{aligned}
\frac{-10x^7y^{20}}{(5x^3y^{10})^2} &= \frac{-10x^7y^{20}}{5^2(x^3)^2(y^{10})^2} && \\
&= \frac{-10x^7y^{20}}{5^2x^{3\cdot 2}y^{10\cdot 2}} && \text{product rule} \\
&= \frac{-10x^7y^{20}}{25x^6y^{20}} && \\
&= -\frac{2}{5}x^{7-6}y^{20-20} && \text{quotient rule} \\
&= -\frac{2}{5}x^1y^0 && \\
&= -\frac{2}{5}x &&
\end{aligned}
$$

ANSWER: $-\frac{2}{5}x$

EXAMPLE 8.1.4. Simplify and state the answer as an expression with positive exponents:

$$\left(\frac{x^5}{2y^2}\right)^{-3}$$

SOLUTION: Get rid of the negative exponent −3 by taking the reciprocal of its base:

$$\left(\frac{x^5}{2y^2}\right)^{-3} = \left(\frac{2y^2}{x^5}\right)^{3}$$

And now distribute exponent over the fraction:

$$\begin{aligned}\left(\frac{2y^2}{x^5}\right)^{3} &= \frac{2^3(y^2)^3}{(x^5)^3} \\ &= \frac{8y^{2\cdot 3}}{x^{5\cdot 3}} \\ &= \frac{8y^6}{x^{15}}\end{aligned}$$

ANSWER: $\frac{8y^6}{x^{15}}$

8.2. Scientific Notation. *Scientific notation* is a way of writing down extremely large and extremely small numbers, and is useful in areas such as physics, engineering, and computer science, to name a few. In fact, many calculators support some sort of scientific notation in order to display very large and very small numbers on a tiny display only a dozen or so characters wide.

A lot of quantities found in nature look bizarre, to say the least, when we express them in familiar units suitable for things we do every day as humans. For example, the mass of our Sun is approximately

$$1988550000000000000000000000000 \quad \text{kg}$$

while the mass of an electron, which is one of the basic building blocks of the tangible world around us, is approximately

$$0.000000000000000000000000000000910938356 \quad \text{kg}$$

The difference between these two quantities is about 60 decimal places, and these are not even the most extreme examples of ridiculously large/small numbers commonly occurring in nature. *Scientific notation* allows writing these down in a concise form, making it easier to understand the true magnitude of a number at a glance.

DEFINITION 8.2.1. A number is said to be in *scientific notation* if it is written in the form

$$D.ddd\ldots \times 10^n$$

where D is a non-zero digit, $ddd\ldots$ are the rest of digits, $\times$ is a multiplication sign, and n is an integer exponent. The factor 10^0, corresponding to $n = 0$, does not have to be shown.

BASIC EXAMPLE 8.2.1. Let's write some numbers in scientific notation. Here's the mass of the Sun in kilograms:

$$1.98855 \times 10^{30}$$

The mass of an electron in kilograms:

$$9.10938356 \times 10^{-31}$$

Recall that multiplying a decimal representation by 10^n amounts to shifting the decimal point by n digits. If n is positive, the decimal point shifts to the right, and if n is negative, the decimal point shifts to the left. Here are some other numbers in decimal notation, which we rewrite in scientific notation:

$$\begin{aligned}
31 &= 3.1 \times 10^1 \\
0.6 &= 6 \times 10^{-1} \\
402 &= 4.02 \times 10^2 \\
0.073 &= 7.3 \times 10^{-2} \\
0.0000101 &= 1.01 \times 10^{-5} \\
12345678 &= 1.2345678 \times 10^7 \\
1450.67 &= 1.45067 \times 10^3
\end{aligned}$$

EXAMPLE 8.2.1. Simplify the expression and state the answer in scientific notation:

$$\frac{16}{10000}$$

SOLUTION:

$$\begin{aligned}
\frac{16}{10000} &= 0.0016 \\
&= 1.6 \times 10^{-3}
\end{aligned}$$

ANSWER: 1.6×10^{-3}

EXAMPLE 8.2.2. Simplify the expression and state the answer in scientific notation:

$$(3.4 \times 10^5)(5 \times 10^7)$$

SOLUTION: The order of multiplication in a product does not matter, so let's multiply the numbers and combine the exponential expressions first:

$$\begin{aligned} (3.4 \times 10^5)(5 \times 10^7) &= (3.4)(5)(10^5)(10^7) \\ &= 17 \cdot 10^{5+7} && \text{product rule} \\ &= 17 \cdot 10^{12} \end{aligned}$$

This is not yet scientific notation; we still need to express 17 as 1.7 times a power of 10:

$$\begin{aligned} 17 \cdot 10^{12} &= (1.7 \times 10^1)(10^{12}) && \text{put 17 in scientific notation} \\ &= 1.7 \times 10^{1+12} && \text{simplify powers of 10} \\ &= 1.7 \times 10^{13} \end{aligned}$$

ANSWER: 1.7×10^{13}

EXAMPLE 8.2.3. Simplify the expression and state the answer in scientific notation:

$$\frac{7.08 \times 10^{-18}}{9.6 \times 10^{-12}}$$

SOLUTION:

$$\begin{aligned} \frac{7.08 \times 10^{-18}}{9.6 \times 10^{-12}} &= \frac{7.08}{9.6} \cdot \frac{10^{-18}}{10^{-12}} \\ &= 0.7375 \cdot 10^{-18-(-12)} && \text{quotient rule} \\ &= 0.7375 \cdot 10^{-6} \\ &= 7.375 \cdot 10^{-1} \cdot 10^{-6} && \text{put 0.7375 in scientific notation} \\ &= 7.375 \cdot 10^{-1+(-6)} && \text{product rule} \\ &= 7.375 \times 10^{-7} \end{aligned}$$

ANSWER: 7.375×10^{-7}

Homework 5.8.

Simplify the given expression and state the answer as an expression with positive exponents.

1. 3^{-3}

2. 10^{-4}

3. $(-3)^{-6}$

4. $(-2)^{-4}$

5. $8x^{-3}$

6. ab^{-5}

7. $\dfrac{4}{x^{-2}}$

8. $\dfrac{Q}{X^{-9}}$

9. $\left(\dfrac{b}{4}\right)^{-2}$

10. $\left(\dfrac{-2}{y}\right)^{-3}$

11. $3^{-5} \cdot 3^{8}$

12. $5^{11} \cdot 5^{-7}$

13. $2x^{-14} \cdot x^{-20}$

14. $-y^{7} \cdot y^{-10}$

15. $(5x^{-2}y^{-3})(2x^{-4}y)$

16. $(3x^{-5}y^{-7})(2xy^{-2})$

17. $(a^{-5})^{3}$

18. $(b^{10})^{-5}$

19. $(2^{-3})^{-2}$

20. $(b^{-5})^{-7}$

21. $(4a^{7})^{-3}$

22. $(7x^{4})^{-2}$

23. $(-2x^{2}y^{-1})^{10}$

24. $(-3a^{-9}b^{4})^{-3}$

25. $\dfrac{15x^{-7}}{10x^{-10}}$

26. $\dfrac{-12y^{-5}}{2y^{-8}}$

27. $\left(\dfrac{x^{4}}{3}\right)^{-2}$

28. $\left(\dfrac{2}{y^{2}}\right)^{-3}$

29. $\left(x^{4}+x^{2}+1\right)^{0}$

30. $\left(17x^{3}y^{-7}\right)^{1}$

31. $\dfrac{-3a^{3}b^{-5}}{-6a^{7}b^{-8}}$

32. $\dfrac{3x^{-2}y^{4}}{-12xy^{-7}}$

33. $(x^{2}y^{-3})(xy^{-5})^{-2}$

34. $(a^{-5}b^{4})(a^{2}b^{-1})^{-3}$

35. $\left(\dfrac{-2x^{5}y^{-6}}{x^{-2}}\right)^{-4}$

36. $\left(\dfrac{3a^{-2}b^{4}}{2b^{-1}}\right)^{-3}$

Simplify the given expression and state the answer in decimal notation.

37. 4.93×10^3

38. 8.14×10^4

39. 8.93×10^{-3}

40. 7.27×10^{-4}

41. 3.007×10^{-6}

42. 1.0204×10^{-5}

43. 5.033×10^5

44. 9.99887×10^6

Write the given number in scientific notation.

45. 15

46. 123

47. 0.7

48. 0.04

49. 37000000

50. 402000

51. 0.0584

52. 0.000695

53. 0.0000007

54. 0.000000065

55. 50000200

56. 4000000000

Simplify the given expression and state the answer in scientific notation.

57. $(3 \times 10^5)(2 \times 10^8)$

58. $(3.2 \times 10^7)(2.2 \times 10^{-4})$

59. $(3.8 \times 10^9)(6.5 \times 10^{-2})$

60. $(7.3 \times 10^7)(8.5 \times 10^{-5})$

61. $\dfrac{8.5 \times 10^8}{3.4 \times 10^{-5}}$

62. $\dfrac{5.7 \times 10^{-2}}{2.4 \times 10^5}$

63. $\dfrac{7.0 \times 10^3}{5.0 \times 10^8}$

64. $\dfrac{7.5 \times 10^{-3}}{8.0 \times 10^{-6}}$

65. $\dfrac{12}{4000}$

66. $\dfrac{13}{5000}$

67. $\dfrac{658.7}{14}$

68. $\dfrac{333}{2500}$

Homework 5.8 Answers.

1. $\dfrac{1}{27}$

3. $\dfrac{1}{729}$

5. $\dfrac{8}{x^3}$

7. $4x^2$

9. $\dfrac{16}{b^2}$

11. 27

13. $\dfrac{2}{x^{34}}$

15. $\dfrac{10}{x^6y^2}$

17. $\dfrac{1}{a^{15}}$

19. 64

21. $\dfrac{1}{64a^{21}}$

23. $\dfrac{1024x^{20}}{y^{10}}$

25. $\dfrac{3}{2}x^3$

27. $\dfrac{9}{x^8}$

29. 1

31. $\dfrac{b^3}{2a^4}$

33. y^7

35. $\dfrac{y^{24}}{16x^{28}}$

37. 4930

39. 0.00893

41. 0.000003007

43. 503300

45. 1.5×10

47. 7×10^{-1}

49. 3.7×10^7

51. 5.84×10^{-2}

53. 7.0×10^{-7}

55. 5.00002×10^7

57. 6×10^{13}

59. 2.47×10^8

61. 2.5×10^{13}

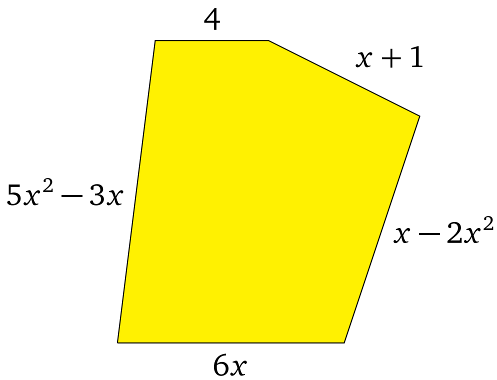
63. 1.4×10^{-5}

65. 3.0×10^{-3}

67. 4.705×10^1

Practice Test 5

1. Find an expression for the perimeter of the shown shape, and simplify it by combining the like terms:



2. Simplify the expression:

$$(3k^5m^0)^2 \cdot (-k^4m^3)$$

3. Simplify the expression:

$$ab^2\left(\frac{a^7b^2}{a^6b^2}\right)^3$$

4. Find the degree of the polynomial:

$$x^{10}-6x^9y^2+7x^6y^6$$

5. Describe the polynomial degree and the number of terms in words:

$$-4x^2y+7xy$$

6. Describe the polynomial degree and the number of terms in words:

$$ax+at-it$$

7. Multiply and simplify by combining the like terms:

$$(3x-6)(4-x+6x^2)$$

8. Simplify the expression by performing the polynomial long division:

$$\frac{14x^3+7x^2-19x+3}{2x+3}$$

9. Simplify the expression by performing the polynomial long division:

$$\frac{6x^3-2x-5}{x-2}$$

10. Simplify the expression and state the answer with positive exponents only:

$$(-5x^{-2}y)(-2x^{-3}y^2)$$

11. Simplify the expression and state the answer with positive exponents only:

$$\left(\frac{x^{-1}y^2}{-5x^{-2}y^4}\right)^{-2}$$

12. Simplify the expression and state the answer in decimal notation:

$$(1.5\times10^{-4})(1.7\times10^7)$$

13. Simplify the expression and state the answer in scientific notation:

$$(4\times10^9)(6\times10^4)$$

14. Simplify the expression and state the answer in scientific notation:

$$\frac{0.9}{1800}$$

Practice Test 5 Answers.

1. $3x^2+5x+5$

2. $-9k^{14}m^3$

3. a^4b^2

4. 11

5. cubic binomial

6. quadratic trinomial

7. $18x^3-39x^2+18x-24$

8. $7x^2-7x+1$

9. $6x^2+12x+22+\dfrac{39}{x-2}$

10. $\dfrac{10y^3}{x^5}$

11. $\dfrac{25y^4}{x^2}$

12. 2250

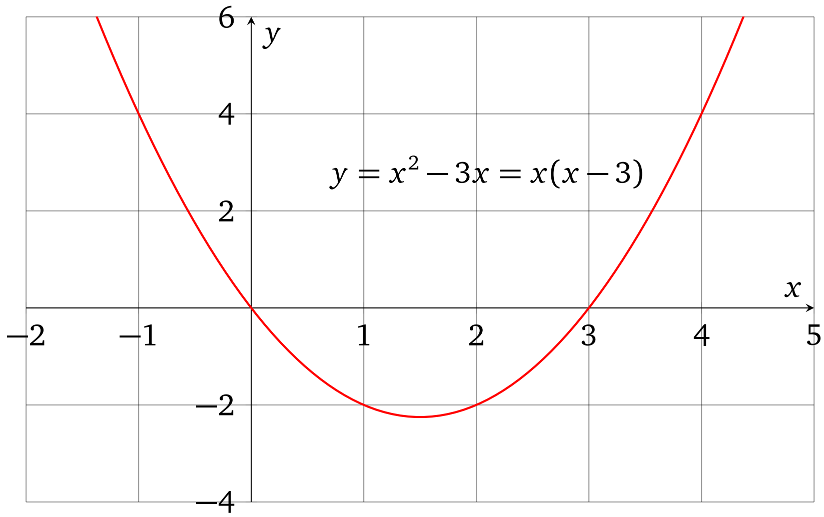
13. 2.4×10^{14}

14. 5.0×10^{-4}

CHAPTER 6

Factoring Polynomials

1. Greatest Common Factor



1.1. Factoring Polynomials. Factoring a polynomial means rewriting it in a product form, with as many polynomial factors as possible. This is useful for solving polynomial equations, as well as for providing an expression which offers hints about the behavior of the corresponding graph. Some polynomials can only be written as a product with a single factor, for example

$$(x+1)$$

Other polynomials can be written with multiple factors:

$$(x^2 - 3x) = (x)(x-3)$$

In this section we start exploring basic strategies for finding these factorizations.

DEFINITION 1.1.1. We call a polynomial *irreducible over reals* if it cannot be written as a product of two or more non-constant polynomials with real coefficients.

We call a polynomial *irreducible over integers* or simply *prime* if it cannot be written as a product of two or more non-constant polynomials with integer coefficients.

THEOREM 1.1.1 (Polynomial Factorization). Every non-constant polynomial expression has a *factorization*: an equivalent form which is a product of polynomial expressions which are *irreducible over reals*. This factorization is unique if we disregard the order of multiplication and state polynomial factors in some kind of standard form.

Stated in a modern way, all polynomial structures are unique factorization domains.

THEOREM 1.1.2. All linear polynomials are irreducible over reals, and hence also prime.

BASIC EXAMPLE 1.1.1. A quadratic polynomial

$$a^2 - b^2$$

can be written as a product of two linear factors

$$(a+b)(a-b)$$

which are irreducible. The only other equivalent polynomial products with two factors are $(a-b)(a+b)$, $(a-b)(b+a)$, and so on, differing only in the order of operations, which is what we mean when we say that the factorization is essentially unique.

In the sections up ahead we will not worry too much about doing a perfect job and obtaining the optimal factorization, but eventually we will list the most notorious types of irreducible polynomials in order to put together a comprehensive factoring strategy.

1.2. Greatest Common Factor.

DEFINITION 1.2.1. The *greatest common factor*, or *GCF* for short, for non-zero integers a and b is the largest positive integer c which is a factor of both a and b. In the absence of common prime factors, the GCF is defined to be 1. The definition extends naturally to larger collections of integers.

Numbers with GCF 1 are called *relatively prime*.

BASIC EXAMPLE 1.2.1. Some basic cases:

integers	**GCF**	
$2,3$	1	no common prime factors
$10,30$	10	the largest common integer factor is 10
$4,6$	2	the only common factor is 2
$6,9,15$	3	the only common factor is 3
$12,18,60$	3	the largest common integer factor is 6

EXAMPLE 1.2.1. Find the GCF for 16 and 20.

SOLUTION: For smaller numbers it's easy enough to guess the GCF by simply staring at them, and you may have figured out already that the GCF in this example is 4. Regardless, here we will show how one can compute GCF even in difficult cases without guessing. First, we represent the numbers as products of primes:

$$\begin{aligned} 16 &= \mathbf{2} \cdot \mathbf{2} \cdot 2 \cdot 2 \\ 20 &= \mathbf{2} \cdot \mathbf{2} \cdot 5 \end{aligned}$$

The GCF is a product of primes which is in common to all of these factorizations, in this case $2 \cdot 2 = 4$.

ANSWER: 4

EXAMPLE 1.2.2. Find the GCF for 15 and 28.

SOLUTION: Represent the numbers as products of primes and detect all the ones they have in common:

$$\begin{aligned} 15 &= 3 \cdot 5 \\ 28 &= 2 \cdot 2 \cdot 7 \end{aligned}$$

These factorizations have no primes in common, meaning 15 and 28 do not have a single common prime factor, so the GCF is 1.

ANSWER: 1

EXAMPLE 1.2.3. Find the GCF for 96 and 66.

SOLUTION: Represent the numbers as products of primes and detect all the ones they have in common:

$$\begin{aligned} 96 &= \mathbf{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot \mathbf{3} \\ 66 &= \mathbf{2} \cdot \mathbf{3} \cdot 11 \end{aligned}$$

The largest collection of prime factors in common is $2 \cdot 3 = 6$

ANSWER: 6

EXAMPLE 1.2.4. Find the GCF for 500 and 120.

SOLUTION: Represent the numbers as products of primes and detect all the ones they have in common:

$$\begin{aligned} 500 &= \mathbf{2}\cdot\mathbf{2}\cdot\mathbf{5}\cdot 5\cdot 5 \\ 120 &= \mathbf{2}\cdot\mathbf{2}\cdot 2\cdot 3\cdot\mathbf{5} \end{aligned}$$

The largest collection of prime factors in common is $2\cdot 2\cdot 5 = 20$

ANSWER: 20

EXAMPLE 1.2.5. Find the GCF for 144, 60, and 90.

SOLUTION: Represent the numbers as products of primes and detect all the ones they have in common:

$$\begin{aligned} 144 &= \mathbf{2}\cdot 2\cdot 2\cdot 2\cdot\mathbf{3}\cdot 3 \\ 60 &= \mathbf{2}\cdot 2\cdot\mathbf{3}\cdot 5 \\ 90 &= \mathbf{2}\cdot\mathbf{3}\cdot 3\cdot 5 \end{aligned}$$

The largest collection of prime factors in common is $2\cdot 3 = 6$

ANSWER: 6

1.3. Monomial GCF.

DEFINITION 1.3.1. The *greatest common factor*, or *GCF* for short, for non-zero monomials A and B is the monomial C such that its coefficient is the GCF of the coefficients of A and B, and the exponent of each variable factor of C is no greater than the exponent of the same variable in either A or B. The definition extends naturally to larger collections of monomials.

Monomials with GCF 1 are called *relatively prime*.

EXAMPLE 1.3.1. Find the GCF for $4a^2b$ and $2a^4b$.

SOLUTION: The GCF for coefficients is 2, the lowest exponent on a is 2, and the lowest exponent on b is 1, so GCF is $2a^2b$.

ANSWER: $2a^2b$

EXAMPLE 1.3.2. Find the GCF for $10x^3y^5$ and $15x^4y^2z^7$.

SOLUTION: The GCF for coefficients is 5, the lowest exponent on x is 3, the lowest exponent on y is 2, and the lowest exponent on z is taken to be 0, since the first monomial does not have z at all. The overall GCF is $5x^3y^2$.

ANSWER: $5x^3y^2$

EXAMPLE 1.3.3. Find the GCF for $14x^7$, $21x^5$, and $6x^4$.

SOLUTION: The GCF for coefficients is 1, and the lowest exponent on x is 4, so GCF is x^4.

ANSWER: x^4

EXAMPLE 1.3.4. Rewrite the expression by factoring out the GCF:

$$12a^2 - 6a$$

SOLUTION: We determine that the GCF is $6a$, and use the distributive property:

$$12a^2 - 6a = 6a(2a - 1)$$

We can check the answer by multiplying the polynomials:

$$6a(2a - 1) = 6a \cdot 2a - 6a \cdot 1 = 12a^2 - 6a$$

ANSWER: $6a(2a - 1)$

EXAMPLE 1.3.5. Rewrite the expression by factoring out the GCF:

$$80x^2y + 40xy - 90xy^3$$

SOLUTION: The GCF for coefficients is 10, the lowest exponent on x is 1, and the lowest exponent on y is 1, so the GCF is $10xy$.

$$80x^2y + 40xy - 90xy^3 = 10xy(8x + 4 - 9y^2)$$

We can check the answer by multiplying the polynomials:

$$\begin{aligned} 10xy(8x+4-9y^2) &= 10xy\cdot 8x+10xy\cdot 4-10xy\cdot 9y^2 \\ &= 80x^2y+40xy-90xy^3 \end{aligned}$$

ANSWER: $10xy(8x+4-9y^2)$

Homework 6.1.

Find the GCF for given expressions.

1. 18, 24

2. 20, 32

3. 12, 30, 60

4. 14, 42, 70

5. $14x$, $7xy^2$

6. $25y^2$, $16y^2z$

7. a^2b^3, $2ab^4$

8. $6x^3y^3$, $4xy^2$

9. $10x$, $2x^2$, $14x^3$

10. $5y^2$, $25y^3$, y^5

Factor out the GCF.

11. $6x+24$

12. $10x+40$

13. $2a^2-2a-8$

14. $12x^2+6x-30$

15. $5t^2+t$

16. $3x^3-x^2$

17. $-10y^2-5y$

18. $-4a^2+3a^3$

19. $6a^6+9a^4-12a^2$

20. $-10x^2-15x^4-9x^3$

21. $5xy^2-10x^2y$

22. $20a^2b^2+15ab$

23. $12x^2y+6xy-2xy^2$

24. $-4a^2-2ab+8b^2$

25. $3x^6y^2+3x^5y^3$

26. $a^4y^5-17a^6y$

Homework 6.1 Answers.

1. 6

3. 6

5. $7x$

7. ab^3

9. $2x$

11. $6(x+4)$

13. $2(a^2-a-4)$

15. $t(5t+1)$

17. $5y(-2y-1)$

19. $3a^2(2a^4+3a^2-4)$

21. $5xy(y-2x)$

23. $2xy(6x+3-y)$

25. $3x^5y^2(x+y)$

2. Grouping

2.1. Factoring by Grouping. An occasional polynomial with 4 terms can be factored using a technique knows as *grouping*, which will be explained in examples.

EXAMPLE 2.1.1. Factor by grouping: $6a + 3b + 10a^2 + 5ab$

SOLUTION: Represent the given polynomial with 4 terms as a sum of two binomials:

$$6a + 3b + 10a^2 + 5ab = (6a + 3b) + (10a^2 + 5ab)$$

For each binomial, factor out the GCF:

$$(6a + 3b) + (10a^2 + 5ab) = 3(2a + b) + 5a(2a + b)$$

Notice that now we have a sum of two terms, and there is a common factor $(2a + b)$. The distributive property allows us to factor it out:

$$3(2a + b) + 5a(2a + b) = (3 + 5a)(2a + b)$$

ANSWER: $(3 + 5a)(2a + b)$

Sometimes it may be hard to see how we got from

$$3(2a + b) + 5a(2a + b)$$

to

$$(3 + 5a)(2a + b)$$

with one application of the distributive property, but we can break this process down into several steps. To see these expressions are equivalent, it may be helpful to create a temporary variable T and set it equal to $2a + b$. Then we can write

$$\begin{aligned} 3(2a + b) + 5a(2a + b) &= 3(T) + 5a(T) && \text{substitute } T \text{ for } 2a + b \\ &= (3 + 5a)T && \text{distributivity} \\ &= (3 + 5a)(2a + b) && \text{substitute } 2a + b \text{ for } T \end{aligned}$$

EXAMPLE 2.1.2. Factor by grouping: $x^5 - x^4 + x^3 - x^2$

SOLUTION: Represent the given polynomial with 4 terms as a sum of two binomials:

$$x^5 - x^4 + x^3 - x^2 = (x^5 - x^4) + (x^3 - x^2)$$

For each binomial, factor out the GCF:

$$(x^5 - x^4) + (x^3 - x^2) = x^4(x - 1) + x^2(x - 1)$$

Notice that now we have a sum of two terms, and there is a common factor $(x-1)$. The distributive property allows us to factor it out:

$$x^4(x-1)+x^2(x-1)=(x^4+x^2)(x-1)$$

ANSWER: $(x^4+x^2)(x-1)$

EXAMPLE 2.1.3. Factor by grouping: $5ab-b+5a-1$

SOLUTION: Represent the given polynomial with 4 terms as a sum of two binomials:

$$5ab-b+5a-1=(5ab-b)+(5a-1)$$

For each binomial, factor out the GCF:

$$(5ab-b)+(5a-1)=b(5a-1)+1(5a-1)$$

Notice that we explicitly factored out the GCF 1 in the second group. This is merely a precaution to make the application of the distributive property more clear:

$$b(5a-1)+1(5a-1)=(b+1)(5a-1)$$

ANSWER: $(b+1)(5a-1)$

EXAMPLE 2.1.4. Factor by grouping: $16a^3-12a^2-12a+9$

SOLUTION: Represent the given polynomial with 4 terms as a sum of two binomials:

$$16a^3-12a^2-12a+9=(16a^3-12a^2)+(-12a+9)$$

For each binomial, factor out the GCF:

$$(16a^3-12a^2)+(-12a+9)=4a^2(4a-3)+3(-4a+3)$$

We have a sum of two terms, but there is no common factor. Fortunately, $(4a-3)$ is the opposite of $(-4a+3)$, so we can obtain a common factor by factoring out -1:

$$4a^2(4a-3)+3(-4a+3)=4a^2(4a-3)-3(4a-3)$$

The distributive property allows us to factor out $(4a-3)$:

$$4a^2(4a-3)-3(4a-3)=(4a^2-3)(4a-3)$$

ANSWER: $(4a^2-3)(4a-3)$

Homework 6.2.

Factor by grouping.

1. $x^3 - 2x^2 + 5x - 10$

2. $z^3 - 3z^2 + 7z - 21$

3. $9n^3 + 6n^2 + 3n + 2$

4. $10x^3 + 25x^2 + 2x + 5$

5. $4t^3 + 20t^2 + 3t + 15$

6. $8a^3 + 2a^2 + 12a + 3$

7. $7x^3 - 5x^2 + 7x - 5$

8. $2t^3 - 12t^2 - t + 6$

9. $5x^3 + 6x^2 - 10x - 12$

10. $10x^3 + 8x^2 - 5x - 4$

11. $x^3 - 3x^2 + 6 - 2x$

12. $a^3 + 8a^2 + 2a + 16$

13. $35x^3 - 10x^2 - 56x + 16$

14. $14v^3 + 10v^2 - 7v - 5$

15. $6x^3 - 48x^2 + 5x - 40$

16. $28p^3 + 21p^2 + 20p + 15$

17. $15ab - 6a + 5b^3 - 2b^2$

18. $3mn - 8m + 15n - 40$

19. $5mn + 2m - 25n - 10$

20. $8xy + 56x - y - 7$

21. $24xy - 30y^3 - 20x + 25y^2$

22. $56ab - 49a - 16b + 14$

23. $10xy + 25x + 12y + 30$

24. $16xy - 6x^2 + 8y - 3x$

Homework 6.2 Answers.

1. $(x-2)(x^2+5)$

3. $(3n+2)(3n^2+1)$

5. $(t+5)(4t^2+3)$

7. $(7x-5)(x^2+1)$

9. $(x^2-2)(5x+6)$

11. $(x-3)(x^2-2)$

13. $(5x^2-8)(7x-2)$

15. $(6x^2+5)(x-8)$

17. $(3a+b^2)(5b-2)$

19. $(m-5)(5n+2)$

21. $(6y-5)(4x-5y^2)$

23. $(5x+6)(2y+5)$

3. Factoring $x^2 + bx + c$

3.1. Guessing the Coefficients.

THEOREM 3.1.1. If a quadratic trinomial $x^2 + bx + c$ with integer coefficients b and c factors over integers at all, then it can be written in the form

$$x^2 + bx + c = (x + m)(x + n)$$

where m and n are integers such that $mn = c$ and $m + n = b$.

PROOF.

$$\begin{aligned}
(x + m)(x + n) &= x(x + n) + m(x + n) \\
&= x^2 + xn + mx + mn \\
&= x^2 + (m + n)x + mn \\
&= x^2 + bx + c
\end{aligned}$$

□

EXAMPLE 3.1.1. Factor the trinomial $x^2 + 5x + 6$

SOLUTION: We need to guess coefficients m and n such that $mn = 6$ and $m + n = 5$. One methodical way to do so is by checking all the possible ways to write 6 as a product of two integer factors:

$$\begin{aligned}
6 &= 1 \cdot 6 \\
&= 2 \cdot 3 \\
&= (-1)(-6) \\
&= (-2)(-3)
\end{aligned}$$

For each one of these, we can check whether $m + n = 5$. If it is, then we have found a factorization.

If $mn = 1 \cdot 6$ then $m + n = 1 + 6 = 7 \neq 5$, no luck.

If $mn = 2 \cdot 3$ then $m + n = 2 + 3 = 5$, success, so $x^2 + 5x + 6 = (x + 2)(x + 3)$.

We can check the answer by multiplying the polynomials:

$$\begin{aligned}
(x + 2)(x + 3) &= x^2 + 3x + 2x + 2 \cdot 3 \\
&= x^2 + 5x + 6
\end{aligned}$$

ANSWER: $(x + 2)(x + 3)$

EXAMPLE 3.1.2. Factor the trinomial $x^2 + 2x - 35$

SOLUTION: We need to guess coefficients m and n such that $mn = -35$ and $m + n = 2$.

$$\begin{aligned} -35 &= 1(-35) \\ &= 5(-7) \\ &= 7(-5) \\ &= 35(-1) \end{aligned}$$

If $mn = 1(-35)$ then $m + n = 1 - 35 = -34 \neq 2$, no luck.

If $mn = 5(-7)$ then $m + n = 5 - 7 = -2 \neq 2$, no luck.

If $mn = 7(-5)$ then $m + n = 7 - 5 = 2$, success, so $x^2 + 2x - 35 = (x + 7)(x - 5)$.

Checking the answer: $(x + 7)(x - 5) = x^2 - 5x + 7x - 35 = x^2 + 2x - 35$.

ANSWER: $(x + 7)(x - 5)$

EXAMPLE 3.1.3. Factor the trinomial $x^2 - 8x + 7$

SOLUTION: We need to guess coefficients m and n such that $mn = 7$ and $m + n = -8$.

$$\begin{aligned} -7 &= 1 \cdot 7 \\ &= (-1)(-7) \end{aligned}$$

If $mn = 1 \cdot 7$ then $m + n = 1 + 7 = 8 \neq -8$, no luck.

If $mn = (-1)(-7)$ then $m + n = -1 + (-7) = -8$, success, so $x^2 - 8x + 7 = (x - 1)(x - 7)$.

ANSWER: $(x - 1)(x - 7)$

EXAMPLE 3.1.4. Factor the trinomial $h^2 - 5h - 2$

SOLUTION: We need to guess coefficients m and n such that $mn = -2$ and $m + n = -5$.

$$\begin{aligned} -2 &= 1(-2) \\ &= 2(-1) \end{aligned}$$

If $mn = 1(-2)$ then $m + n = 1 - 2 = -1 \neq -5$, no luck.

If $mn = 2(-1)$ then $m + n = 2 - 1 = 1 \neq -5$, no luck.

There are essentially no other ways to have two integers with a product -2, we tried every possible one. Since both addition and multiplication are commutative, checking the products $(-1)2$ and $-2(1)$ would be redundant. We can conclude that the given polynomial is irreducible over integers, or prime for short.

ANSWER: prime

3.2. Variations of the Pattern. It is possible to spot this binomial pattern in other places as well, like quadratics with leading coefficient -1, or the ones with a constant coefficient ± 1, or multivariate polynomials.

EXAMPLE 3.2.1. Factor the trinomial $-x^2 + 4x + 21$

SOLUTION: This is not a trinomial of the form $x^2 + bx + c$ because the leading coefficient is -1 when it needs to be 1. But we can easily fix that by factoring out -1:

$$-x^2 + 4x + 21 \quad = \quad -(x^2 - 4x - 21)$$

Now we need to find two numbers m and n such that $mn = -21$ and $m + n = -4$. Let us try $m = 3$ and $n = -7$:

$$\begin{aligned} -(x+3)(x-7) &= -(x^2 - 7x + 3x - 21) \\ &= -(x^2 - 4x - 21) \qquad \text{looks right} \\ &= -x^2 + 4x + 21 \end{aligned}$$

Note that there are other ways to write this factorization, which look different, but are essentially the same. They can be obtained by distributing the leading minus over either sum:

$$-(x+3)(x-7) = (-x-3)(x-7) = (x+3)(-x+7)$$

ANSWER: $-(x+3)(x-7)$

EXAMPLE 3.2.2. Factor the trinomial $-6x^2 - x + 1$

SOLUTION: This is not really a polynomial of the form $x^2 + bx + c$, since the leading coefficient is -6 when it needs to be 1. We can, however, notice that the constant term is 1, and if we expect an answer in the form

$$(mx + 1)(nx + 1)$$

then we are looking for m and n such that $mn=-6$ and $m+n=-1$.

If $mn=1(-6)$ then $m+n=1-6=-5$, no luck.

If $mn=2(-3)$ then $m+n=2-3=-1$, success, so $-6x^2-x+1=(2x+1)(-3x+1)$. We can check the answer by multiplying the polynomials:

$$\begin{aligned}(2x+1)(-3x+1) &= (2x)(-3x)+(2x)1+1(-3x)+1\cdot 1\\ &= -6x^2+2x-3x+1\\ &= -6x^2-x+1\end{aligned}$$

ANSWER: $(2x+1)(-3x+1)$

EXAMPLE 3.2.3. Factor the trinomial $y^2+11yz-26z^2$

SOLUTION: This is a trinomial with two variables, but it uses the same pattern for coefficients as x^2+bx+c, with the mixed term $11yz$ playing the role of the linear term bx, and the $-26z^2$ term playing the role of the constant c. So we are looking for m and n such that $mn=-26$ and $m+n=11$.

If $mn=1(-26)$ then $m+n=1-26=-25\neq 11$, no luck.

If $mn=2(-13)$ then $m+n=2-13=-11\neq 11$, no luck.

If $mn=13(-2)$ then $m+n=13-2=11$, success, so $y^2+11yz-26z^2=(y+13z)(y-2z)$.

ANSWER: $(y+13z)(y-2z)$

3.3. Basic Factoring Strategy. By now we have seen 3 different techniques for factoring a polynomial expression: factoring out the GCF, factoring by grouping, and factoring a simple trinomial. Recall that the final factorization of any polynomial is essentially unique, so we can apply these and other techniques incrementally and in any order we want. In most cases, however, it is convenient to follow a simple strategy.

To take the advantage of all the factoring techniques we know so far,

(1) Factor out the GCF, unless it's 1.
(2) If any of the remaining polynomial factors have four terms, try to factor them by grouping.
(3) If any of the remaining polynomial factors are trinomials of the form x^2+bx+c, try to factor them by guessing integer coefficients.

EXAMPLE 3.3.1. Factor the trinomial x^3+3x^2+2x

SOLUTION: Factor out the GCF first:

$$x^3+3x^2+2x = x(x^2+3x+2)$$

We attempt to factor the quadratic trinomial next. We need to replace it by an expression $(x+n)(x+m)$ where $mn=2$ and $m+n=3$. The ways to write 2 as a product of two integers mn:

$$\begin{aligned} 2 &= 1\cdot 2 \\ &= (-1)(-2) \end{aligned}$$

If $mn = 1\cdot 2$ then $m+n = 1+2 = 3$, success, so

$$x(x^2+3x+2) = x(x+2)(x+3)$$

ANSWER: $x(x+2)(x+3)$

EXAMPLE 3.3.2. Factor the polynomial $10x^3y-2x^2y+30xy-6y$

SOLUTION: Every term of this polynomial has factors 2 and y, so we start by factoring out the GCF $2y$:

$$10x^3y-2x^2y+30xy-6y = 2y(5x^3-x^2+15x-3)$$

We attempt to factor the polynomial with 4 terms next. The only appropriate technique at our disposal is grouping. Notice that the GCF will have to be simply rewritten all the way until the answer. Nothing else we factor can possibly affect it.

$$\begin{aligned} 2y(5x^3-x^2+15x-3) &= 2y\left((5x^3-x^2)+(15x-3)\right) && \text{create binomial groups} \\ &= 2y\left(x^2(5x-1)+3(5x-1)\right) && \text{find GCF for each binomial} \\ &= 2y\left((x^2+3)(5x-1)\right) && \text{distributivity} \end{aligned}$$

ANSWER: $2y(x^2+3)(5x-1)$

Homework 6.3.

Factor the polynomial expression.

1. $x^2 - 8x + 16$

2. $a^2 + 9a + 20$

3. $x^2 - 11x + 10$

4. $y^2 - 8y + 7$

5. $t^2 + 9t + 14$

6. $a^2 + 14a + 40$

7. $b^2 + 5b + 4$

8. $z^2 + 8z + 7$

9. $d^2 + 7d + 10$

10. $x^2 + 8x + 15$

11. $a^2 - 2a - 15$

12. $b^2 - b - 42$

13. $x^2 - 4x - 45$

14. $x^2 + 22x + 121$

15. $2x^2 + 14x - 36$

16. $3y^2 + 9y - 84$

17. $x^3 - 6x^2 - 16x$

18. $x^3 - x^2 - 20x$

19. $5b^2 + 35b - 150$

20. $x^4 + x^3 - 56x^2$

21. $-b^5 + b^4 + 2b^3$

22. $-2x^2 + 4x + 70$

23. $a^2 + 5a + 9$

24. $x^2 - 7x + 18$

25. $-3x^3 + 63x^2 + 300x$

26. $2x^2 - 42x - 144$

27. $x^2 + 7xy + 10y^2$

28. $x^2 - 2xy - 3y^2$

29. $b^2 + 7bc + 7c^2$

30. $5x^7 - 20x^6 - 25x^5$

Homework 6.3 Answers.

1. $(x-4)(x-4)$

3. $(x-1)(x-10)$

5. $(t+7)(t+2)$

7. $(b+1)(b+4)$

9. $(d+2)(d+5)$

11. $(a-5)(a+3)$

13. $(x+5)(x-9)$

15. $2(x-2)(x+9)$

17. $x(x+2)(x-8)$

19. $5(b-3)(b+10)$

21. $-b^3(b-2)(b+1)$

23. prime

25. $-3x(x-25)(x+4)$

27. $(x+5y)(x+2y)$

29. prime

4. Factoring $ax^2 + bx + c$

There are many ways to factor a trinomial of the form $ax^2 + bx + c$, and they all involve some amount of guessing, or trial and error. It is important to understand that experienced human algebraists rarely go through a list of possibilities to factor trinomials. Instead, after doing dozens and dozens of basic examples, they train the brain to recognize these patterns instantly, and combinations of coefficients which just *feel right* often turn out to be the solutions. The reader may expect to learn (eventually) to try some combinations of coefficients "at random", check whether the factorization works, and "stumble" on the correct answer after only a few attempts.

4.1. ***ac*** **Method.** The ac method is named after the product of the coefficients a and c, and it works by guessing numbers s and t such that $st = ac$ and $s + t = b$. The method is explained in the following examples, and proven at the end of this subsection. The general course of action is to

(1) Factor out the GCF if possible.
(2) For the resulting trinomial, find s and t with $st = ac$ and $s + t = b$.
(3) Rewrite the trinomial in terms of s and t, then factor it by grouping.

EXAMPLE 4.1.1. Factor the trinomial $2x^2 + 7x + 3$

SOLUTION: Here the GCF for coefficients is 1, so we can proceed with $a = 2$, $b = 7$, and $c = 3$. We look for two numbers s and t with $st = ac = 6$ and $s + t = b = 7$. There are only a few ways to represent 6 as a product of two numbers:

$$1 \cdot 6, \quad 2 \cdot 3, \quad -1 \cdot (-6), \quad -2 \cdot (-3)$$

We can see that $1+6 = 7 = b$, so we let $s = 1$, $t = 6$, and factor the trinomial by grouping:

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x^2 + (1+6)x + 3 \\ &= 2x^2 + 1x + 6x + 3 \\ &= x(2x+1) + 3(2x+1) && \text{factor out GCF from each group} \\ &= (x+3)(2x+1) \end{aligned}$$

ANSWER: $(x+3)(2x+1)$

EXAMPLE 4.1.2. Factor the trinomial $16x^2 + 44x + 30$

SOLUTION: In order for the ac method to work most efficiently, we must begin by factoring out the GCF:

$$16x^2 + 44x + 30 \quad = \quad 2(8x^2 + 22x + 15) \qquad \text{a=8, b=22, c=15}$$

Now that the trinomial coefficients are relatively prime, we have $ac = 8 \cdot 15 = 120$, and we are looking for two factors $st = 120$ which also give us $s + t = 22$. We have several choices:

$$1 \cdot 120, \quad 2 \cdot 60, \quad 3 \cdot 40, \quad 4 \cdot 30, \quad 5 \cdot 24, \quad 6 \cdot 20, \quad 8 \cdot 15, \quad 10 \cdot 12,$$

and the same combinations using negative numbers. One of these works: $10 + 12 = b = 22$, so we use it to factor the trinomial by grouping:

$$\begin{aligned} 8x^2 + 22x + 15 &= 8x^2 + (10 + 12)x + 15 \\ &= 8x^2 + 10x + 12x + 15 \\ &= 2x(4x + 5) + 3(4x + 5) \\ &= (2x + 3)(4x + 5) \end{aligned}$$

The answer is the product of the GCF we found initially and the factored trinomial.

ANSWER: $2(2x + 3)(4x + 5)$

EXAMPLE 4.1.3. Factor the trinomial $5x^2 - 23x - 10$

SOLUTION: Here $a = 5$, $b = -23$, and $c = -10$, so $ac = -50$. We have several choices for s and t:

$$1 \cdot (-50), \quad 50 \cdot (-1), \quad 2 \cdot (-25), \quad 25 \cdot (-2), \quad 5 \cdot (-10), \quad 10 \cdot (-5)$$

One combination works: $2 - 25 = -23 = b$, so we use these coefficients for grouping:

$$\begin{aligned} 5x^2 - 23x - 10 &= 5x^2 + (2 - 25)x - 10 \\ &= 5x^2 + 2x - 25x - 10 \\ &= x(5x + 2) - 5(5x + 2) \\ &= (x - 5)(5x + 2) \end{aligned}$$

ANSWER: $(x - 5)(5x + 2)$

EXAMPLE 4.1.4. Factor the trinomial $4x^2 - 7x + 6$

SOLUTION: Here $ac = 4 \cdot 6 = 24$, and our choices for $st = ca$ are:

$$1 \cdot 24, \quad 2 \cdot 12, \quad 3 \cdot 8, \quad 4 \cdot 6,$$

and the similar combinations with negative numbers. None of these pairs add up to $b = 7$ though, and hence the trinomial is prime.

ANSWER: prime

What remains is to show that the ac method is indeed an effective procedure which never fails to detect a factorization, as long as one exists.

THEOREM 4.1.1 (ac Method). If a quadratic trinomial ax^2+bx+c with integer coefficients a, b and c factors over integers at all, then we can find integers s and t such that $st = ac$ and $s + t = b$, rewrite

$$ax^2 + bx + c \quad = \quad ax^2 + sx + tx + c$$

and then factor by grouping.

PROOF. Assuming that the polynomial is not prime, the fact that numbers s and t can be found with $st = ac$ and $s + t = b$ follows directly from the proof of the theorem (4.3.1). The only interesting part here is in showing that the factoring by grouping is guaranteed to work. So consider the expression

$$\begin{aligned} ax^2 + bx + c &= ax^2 + (s+t)x + c \\ &= ax^2 + sx + tx + c \end{aligned}$$

Let $s/a = q/r$, where q/r is in lowest terms. In other words, r is the product of precisely those factors of a which are "missing" from s. It is important to note that a/r is an integer, but also t/r is an integer, since $st/a = qt/r$ is an integer. Now apply grouping:

$$\begin{aligned} ax^2 + sx + tx + c &= \frac{a}{r}x\left(rx + \frac{rs}{a}\right) + \frac{t}{r}\left(rx + \frac{rs}{a}\right) \\ &= \left(\frac{a}{r}x + \frac{t}{r}\right)\left(rx + \frac{rs}{a}\right) \end{aligned}$$

Note that this is honest grouping: we really are taking out the GCF in both groups. Dividing $rs/a = q$ by any prime factor of r would produce a non-integer. □

4.2. Box Method. The box method is really just a visual aid to the ac method, where we accomplish the grouping stage by filling out a small table, or "box". Just as the ac method, the box method should be applied after factoring out the GCF.

EXAMPLE 4.2.1. Factor the trinomial $6x^2 - 5x - 4$

SOLUTION:

Here we have $a = 6$, $b = -5$, and $c = -4$. The coefficients are relatively prime, so we don't need to worry about the GCF. We start by drawing the "box" and filling it out with the quadratic term ax^2 and the constant term c as follows:

$6x^2$	
	-4

Next we find numbers s and t with $st = ac$ and $s + t = b$, just like when using the ac method, and fill the remaining cells with sx and tx. In this case, we can use 3 and -8 because $3 \cdot (-8) = -24 = ac$ and $3 - 8 = -5 = b$.

$6x^2$	$3x$
$-8x$	-4

Then we find the GCF for each row and each column, using the sign *closest* to it in the box:

	$2x$	1
$3x$	$6x^2$	$3x$
-4	$-8x$	-4

Finally, we write the row GCFs as one binomial factor, and the column GCFs as the other binomial factor:

$$(3x - 4)(2x + 1)$$

As always, we can check that the factoring is correct by multiplying and simplifying.

ANSWER: $(3x - 4)(2x + 1)$

EXAMPLE 4.2.2. Factor the trinomial $15x^3 - 54x^2 + 27x$

SOLUTION:

It helps to start by factoring out the GCF, in this case $3x$:

$$15x^3 - 54x^2 + 27x \quad = \quad 3x(5x^2 - 18x + 9)$$

Now we have a trinomial with $a = 5$, $b = -18$, and $c = 9$. We draw the "box" and fill it out with the quadratic term ax^2 and the constant term c as follows:

$5x^2$	
	9

Next we find numbers s and t with $st = ac$ and $s + t = b$, just like when using the ac method, and fill the remaining cells with sx and tx. In this case, we can use -15 and -3 because $-15 \cdot (-3) = 45 = ac$ and $-15 - 3 = -18 = b$.

$5x^2$	$-15x$
$-3x$	9

Then we find the GCF for each row and each column, using the sign *closest* to it in the box:

	x	-3
$5x$	$5x^2$	$-15x$
-3	$-3x$	-9

Finally, we write the row GCFs as one binomial factor, and the column GCFs as the other binomial factor:

$$(5x - 3)(x - 3)$$

As always, we can check that the factoring is correct by multiplying and simplifying. The answer is the product of the GCF we found initially and the factored trinomial.

ANSWER: $3x(5x - 3)(x - 3)$

4.3. Guessing the Coefficients. The most naive way of factoring a trinomial is by simply guessing all four coefficients in the binomial factors. This is perhaps not as pleasant in practice as the ac or the box methods, but it can be remarkably fast when the coefficients are small and the algebraist is well trained.

THEOREM 4.3.1. If a quadratic trinomial $ax^2 + bx + c$ with integer coefficients a, b and c factors over integers at all, then it can be written in the form

$$ax^2 + bx + c = (gx + m)(hx + n)$$

where g, h, m, n are integers such that $gh = a$, $mn = c$, and $gn + mh = b$.

PROOF.

$$\begin{aligned}(gx+m)(hx+n) &= gx(hx+n)+m(hx+n)\\ &= gxhx+gxn+mhx+mn\\ &= (gh)x^2+(gn+mh)x+mn\\ &= ax^2+bx+c\end{aligned}$$

This proves the statement of the theorem. Observe that as a consequence, we can always expect to find numbers s and t such that $st = ac$ and $s + t = b$. Just let $s = gh$ and let $t = mn$. □

We should also consider that when we try to factor

$$ax^2 + bx + c$$

we would like to make sure we checked *every* possible combination of integer coefficients g, h, m, n such that $gh = a$ and $mn = c$, but we want to avoid redundant checking of equivalent combinations of coefficients. To do so, we will use a specific pattern. We will also assume that a is positive: if it is not, we can easily make it positive by factoring out -1, for example

$$-3x^2 + 5x + 2 = -(3x^2 - 5x - 2)$$

The following theorem allows us to trim the list of all possible combinations of coefficients to just the non-redundant ones.

THEOREM 4.3.2. In order to check whether a trinomial with integer coefficients

$$ax^2 + bx + c$$

with $a > 0$, factors over integers as

$$(gx + m)(hx + n)$$

it is sufficient to examine every pair of *positive* integers g and h such that $gh = a$ and $g \leq h$, and for each one of these, *every possible* integer pair m and n with $mn = c$.

EXAMPLE 4.3.1. Factor the trinomial $5x^2 - 17x + 6$

SOLUTION: We need to guess 4 coefficients g, h, m, n such that $gh = 5$, $mn = 6$, and $gn + mh = -17$. We will do so by going over all of the ways to represent coefficients 5 and 10 as products of two integers. The combinations are presented in the table that follows. Note that for $g \cdot h$ we only try one pair: $1 \cdot 5$, because this is the only way to get $gh = 5$ with positive $g \le h$. But for m and n we try every possible assignment which gives us $mn = 10$.

g	h	m	n	$gn + mh$
1	5	1	6	11
1	5	2	3	13
1	5	3	2	17
1	5	6	1	31
1	5	−1	−6	−11
1	5	−2	−3	−13
1	**5**	**−3**	**−2**	**−17**
1	5	−6	−1	−31

The highlighted row $gn + mh = -17$ shows the correct coefficients, so the answer is

$$(gx + m)(hx + n) = (x - 3)(5x - 2)$$

ANSWER: $(x - 3)(5x - 2)$

Homework 6.4.

1. $2x^2 - 13x - 7$

2. $2x^2 + 7x - 4$

3. $9x^2 - 9x + 2$

4. $5a^2 - 19a - 4$

5. $3x^2 + 19x + 6$

6. $35x^2 + 34x + 8$

7. $8t^2 + 6t - 9$

8. $-4x^2 - 12x - 5$

9. $1 - 4x + 3x^2$

10. $9 - 21x - 18x^2$

11. $2x^2 - 6x - 17$

12. $15 + c - 2c^2$

13. $25x^2 - 40x + 16$

14. $49x^2 - 42x + 9$

15. $-20y^2 + 25y - 5$

16. $-10a^2 + 8a + 18$

17. $6z^2 + 21z + 15$

18. $12a^2 + 68a - 24$

19. $16t^2 - 23t + 7$

20. $9t^2 - 14t + 5$

21. $9x^2 + 18x + 5$

22. $16x^2 + 32x + 7$

23. $18x^3 + 33x^2 + 9x$

24. $6a^3 - 4a^2 - 10a$

25. $14x^4 - 19x^3 - 3x^2$

26. $70y^4 - 68y^3 + 16y^2$

Homework 6.4 Answers.

1. $(2x+1)(x-7)$

3. $(3x-1)(3x-2)$

5. $(3x+1)(x+6)$

7. $(2t+3)(4t-3)$

9. $(3x-1)(x-1)$

11. prime

13. $(5x-4)^2$

15. $-5(y-1)(4y-1)$

17. $3(x+1)(2x+5)$

19. $(16t-7)(t-1)$

21. $(3x+1)(3x+5)$

23. $3x(3x+1)(2x+3)$

25. $x^2(2x-3)(7x+1)$

5. Factoring Special Products

5.1. Difference of Squares. Recall the difference of squares formula:

$$A^2 - B^2 = (A+B)(A-B)$$

for any two reals A and B.

EXAMPLE 5.1.1. Factor the special product expression $25x^2 - 9y^2$

SOLUTION: This can actually be done by guessing coefficients for a factored form of the trinomial $25x^2 + 0xy - 9y^2$, but being able to detect a difference of squares saves a lot of time and effort:

$$\begin{aligned} 25x^2 - 9y^2 &= (5x)^2 - (3y)^2 \\ &= (5x+3y)(5x-3y) \end{aligned}$$

ANSWER: $(5x+3y)(5x-3y)$

EXAMPLE 5.1.2. Factor the special product expression $200x^6 - 18$

SOLUTION: 200 and 18 are not squares of integers, but may be a special product will become evident if we factor out GCF first:

$$\begin{aligned} 200x^6 - 18 &= 2(100x^6 - 9) \\ &= 2\left((10x^3)^2 - (3)^2\right) \\ &= 2\left((10x^3+3)(10x^3-3)\right) \end{aligned}$$

ANSWER: $2(10x^3+3)(10x^3-3)$

It is worthy of note that a sum of squares is irreducible over reals.

THEOREM 5.1.1. If X and Y are monomials with relatively prime coefficients and no variables in common, then the binomial $X^2 + Y^2$ is irreducible over reals.

BASIC EXAMPLE 5.1.1. It may be tempting to write that x^2+4 is equivalent to $(x+2)(x-2)$ or perhaps $(x+2)^2$, but neither statement is true. Just like any other sum of squares, x^2+4 is prime.

EXAMPLE 5.1.3. Factor the expression $27x^7 + 3x^5$.

SOLUTION: Factor out GCF first.

$$27x^7 + 3x^5 = 3x^5(9x^2 + 1)$$

$(9x^2 + 1)$ is a sum of squares, which is prime, so we cannot factor the polynomial any further.

ANSWER: $3x^5(9x^2 + 1)$

EXAMPLE 5.1.4. Factor the expression $a^4 - b^4$

SOLUTION: Factor as a difference of squares:

$$\begin{aligned} a^4 - b^4 &= (a^2)^2 - (b^2)^2 \\ &= (a^2 + b^2)(a^2 - b^2) \end{aligned}$$

The factor $(a^2 + b^2)$ is a sum of squares, which is prime. The factor $(a^2 - b^2)$ though is a difference of squares again, so we can factor it further:

$$(a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)((a + b)(a - b))$$

ANSWER: $(a^2 + b^2)(a + b)(a - b)$

5.2. Square of a Binomial. Recall the binomial square formulas:

$$\begin{aligned} A^2+2AB+B^2 &= (A+B)^2 \\ A^2-2AB+B^2 &= (A-B)^2 \end{aligned}$$

for any two reals A and B.

EXAMPLE 5.2.1. Factor the special product expression $64x^2+16x+1$

SOLUTION: This can be done by guessing the coefficients, but detecting the square of a binomial saves some time. Here we notice that $64x^2=(8x)^2$ and $1=(1)^2$, while $16x$ is indeed $2(8)(x)$.

$$\begin{aligned} 64x^2+16x+1 &= (8x)^2+2(8)(x)+(1)^2 \\ &= (8x+1)^2 \end{aligned}$$

ANSWER: $(8x+1)^2$

EXAMPLE 5.2.2. Factor the special product expression $25y^2-60yz+36z^2$

SOLUTION: This can be done by guessing the coefficients, but detecting the square of a binomial saves some time. Here we notice that $25y^2=(5y)^2$, and $36z^2=(6z)^2$, while $-60yz$ is indeed $-2(5y)(6x)$.

$$\begin{aligned} 25y^2-60yz+36z^2 &= (5y)^2-2(5y)(6x)+(6z)^2 \\ &= (5y-6z)^2 \end{aligned}$$

ANSWER: $(5y-6z)^2$

Homework 6.5.

Factor the given expression.

1. $x^2 - 14x + 49$

2. $x^2 - 10x + 25$

3. $4x^2 + 8x + 4$

4. $9y^2 + 12y + 4$

5. $x^2 - 100$

6. $25x^2 - 1$

7. $3n^3 + 60n^2 + 300n$

8. $x^5 + 24x^4 + 144x^3$

9. $20a^2 + 100a + 125$

10. $27x^2 + 36x + 12$

11. $2x^2 + 28x + 98$

12. $16y^2 - 24y + 9$

13. $6x^2 - 24$

14. $5y^2 - 5$

15. $200t^2 - 8$

16. $8y^2 - 98$

17. $x^2 - 5xy + 9y^2$

18. $a^2 - 9ab - 10b^2$

19. $5x^4 + 125x^2$

20. $ab^2 + 9a$

21. $25y^2 - 4$

22. $80x^2 - 45$

23. $81x^4 - 625$

24. $x^4y^4 - 1$

25. $x^8 - 256$

26. $y^{16} - 1$

27. $x^2 + 16xy + 64y^2$

28. $81y^2 - 18yz + z^2$

29. $32x^2 - 80xy + 50y^2$

30. $-36c^2 - 96cd - 64d^2$

Homework 6.5 Answers.

1. $(x-7)^2$

3. $4(x+1)^2$

5. $(x+10)(x-10)$

7. $3n(n+10)^2$

9. $5(2x+5)^2$

11. $2(x+7)^2$

13. $6(x+2)(x-2)$

15. $8(5t^2+1)(5t^2-1)$

17. prime

19. $5x^2(x^2+25)$

21. $(5y+2)(5y-2)$

23. $(9x^2+25)(3x+5)(3x-5)$

25. $(x^4+16)(x^2+4)(x+2)(x-2)$

27. $(x+8y)^2$

29. $2(4x-5y^2)$

6. General Factoring Strategy

6.1. Irreducible Polynomials. In this section we will combine all of the polynomial factoring techniques presented so far. We will also describe which polynomials are irreducible. Knowing which polynomials cannot be factored any further will gives us an assurance that our answers are in fact in the fully factored form.

(1) By theorem 1.1.2, all linear polynomials are prime.
(2) Some prime trinomials such as x^2+x+1 can be detected by trying out every feasible combination of coefficients in factorizations. See theorems 3.1.1 and 4.3.1.
(3) By theorem 5.1.1, sums of relatively prime monomial squares X^2+Y^2 are irreducible over reals and therefore also prime.

BASIC EXAMPLE 6.1.1. In this text we will usually recognize a fully factored polynomial as a product of linear and irreducible quadratic factors, something like

$$5x^7(x+1)^3(x^2+4)$$

We use exponential notation for brevity, but if we think of this as a product of polynomial factors, then x^7 as a linear factor x raised to the 7th power, $(x+1)^3$ is a linear factor $x+1$ raised to the 3rd power, and x^2+4 is an irreducible quadratic factor raised to the 1st power.

6.2. Factoring Strategy. By now we have seen several different techniques for factoring a polynomial expression: factoring out the GCF, factoring by grouping, factoring a trinomial by guessing the coefficients, and the special product shortcuts. Recall that the final factorization of any polynomial is essentially unique, so we can apply these techniques incrementally and in any order we want. In most cases, however, it is convenient to try them in a specific order.

To take the advantage of all the factoring techniques we know so far,

(1) Factor out the GCF, unless it's 1.
(2) If a polynomial factor has four terms, try to factor it by grouping.
(3) If a polynomial factor has two terms, try to factor it as a difference of squares A^2-B^2.
(4) If a polynomial factor is a trinomial of the form x^2+bx+c, try to factor it by guessing two integer coefficients for a product of two binomials.
(5) If a polynomial factor is a trinomial of the form ax^2+bx+c, try to factor it as a binomial square, and if that fails, try to guess four integer coefficients for a product of two binomials.

One may have to apply these techniques repeatedly, until only prime polynomial factors remain.

EXAMPLE 6.2.1. Factor the polynomial completely:

$$x^2 + x$$

SOLUTION: $x^2 + x = x(x+1)$, which is a product of a linear monomial and a linear binomial, so every factor is irreducible.

ANSWER: $x(x+1)$

EXAMPLE 6.2.2. Factor the polynomial completely:

$$3x^3 - 3x^2 + x - 1$$

SOLUTION: Factoring by grouping yields

$$\begin{aligned} 3x^3 - 3x^2 + x - 1 &= 3x^2(x-1) + 1(x-1) \\ &= (3x^2+1)(x-1) \end{aligned}$$

The factor $3x^2+1$ is an irreducible quadratic binomial, and $x-1$ is linear, so also irreducible over reals.

ANSWER: $(3x^2+1)(x-1)$

EXAMPLE 6.2.3. Factor the polynomial completely:

$$-m^6 + 2m^5 + 35m^4$$

SOLUTION: The GCF is m^4, and we also notice that factoring out -1 will create an easy trinomial pattern:

$$\begin{aligned} -m^6 + 2m^5 + 35m^4 &= -m^4(m^2 - 2m - 35) \\ &= -m^4(m+5)(m-7) \end{aligned}$$

We think of m^4 as a linear factor m raised to 4th power, so every factor is linear, and hence prime, and we are done.

ANSWER: $-m^4(m+5)(m-7)$

EXAMPLE 6.2.4. Factor the polynomial completely:

$$6x^3y^5 - 21x^4y^5 + 3x^3y^6$$

SOLUTION: We start by factoring out the GCF:

$$6x^3y^5 - 21x^4y^5 + 3x^3y^6 \quad = \quad 3x^3y^5(2 - 7x + y)$$

This is a product of a monomial and a linear trinomial, so we are done.

ANSWER: $3x^3y^5(2 - 7x + y)$

EXAMPLE 6.2.5. Factor the polynomial completely:

$$a^3 - 5a^2 - 4a + 20$$

SOLUTION: The GCF is 1, and there are four terms, so let's try grouping:

$$\begin{aligned} a^3 - 5a^2 - 4a + 20 \quad &= \quad a^2(a-5) - 4(a-5) \\ &= \quad (a^2 - 4)(a - 5) \end{aligned}$$

$a^2 - 4$ is not linear, so we should check whether we can factor it more, and in fact we can as a difference of squares:

$$\begin{aligned} (a^2 - 4)(a - 5) \quad &= \quad (a^2 - 2^2)(a - 5) \\ &= \quad ((a+2)(a-2))(a-5) \end{aligned}$$

ANSWER: $(a + 2)(a - 2)(a - 5)$

Homework 6.6.

Factor the given polynomial completely.

1. $5y^2-80$

2. $10y^2-490$

3. $18y^4-12y^3$

4. x^2+x-12

5. $6x^3-9x^2+2x-3$

6. $x^2+5x+xy+5y$

7. $6x^3+x^2-5x$

8. $25y^3-30y^2+9y$

9. $3x^4+6x^2$

10. $4y^2+36$

11. $4z^2w+13zw+10w$

12. $5x^2-30x+10$

13. $x^2y^4+4xy^4-32y^4$

14. $-60+52y-8y^2$

15. $x^4+6x^3-6x^2-36x$

16. $z^5-2z^4+5z^3-10z^2$

17. $5y^5-80y$

18. $7z^4-112$

19. $x^2-5xy+8y^2$

20. $25z^2+10zx+x^2$

21. $16t^2+28t-30$

22. $36x^2+24x-45$

23. $-6x^2-7x+1$

24. $16x^4+y^4$

25. $4w^3+20w^2-4w-20$

26. $u^4+3u^3-16u^2-48u$

27. $4x^2+y^2-4xy$

28. $11a^4-11b^4$

29. $-360y^2+x^2+2xy$

30. $3x^3+17x^2y-6xy^2$

Homework 6.6 Answers.

1. $5(y+4)(y-4)$

3. $6y^3(3y-2)$

5. $(2x-3)(3x^2+1)$

7. $x(6x-5)(x+1)$

9. $3x^2(x^2+2)$

11. $w(4z+5)(z+2)$

13. $y^4(x-4)(x+8)$

15. $x(x^2-6)(x+6)$

17. $5y(y^2+4)(y+2)(y-2)$

19. prime

21. $2(2t+5)(4t-3)$

23. prime

25. $4(w+5)(w+1)(w-1)$

27. $(2x-y)^2$

29. $(x+20y)(x-18y)$

7. Solving Equations by Factoring

7.1. Zero Product Property.

THEOREM 7.1.1. If a product of several numbers is zero, then one of these numbers must be zero. Conversely, if one of the factors in a product is zero, then the value of the product is zero.

Formally, for all real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$. This statement generalizes naturally to products with any number of factors.

THEOREM 7.1.2 (Zero Product in Equations). If P and Q are polynomials, then the solution set of the equation $PQ = 0$ consists of all solutions of $P = 0$ together with all solutions of $Q = 0$.

EXAMPLE 7.1.1. Solve the equation

$$x^2 - 3x + 2 = 0$$

SOLUTION: Factoring the expression on the left yields

$$(x-1)(x-2) = 0$$

Here $(x-1)$ and $(x-2)$ are the polynomial factors P and Q respectively in the theorem 7.1.2.

Solving the equation $x - 1 = 0$ gives us $x = 1$.

Solving the equation $x - 2 = 0$ gives us $x = 2$.

So the solutions to the quadratic equation are 1 and 2.

Let's check the answers. If $x = 1$, then

$$x^2 - 3x + 2 = 1^2 - 3 \cdot 1 + 2 = 0$$

If $x = 2$, then

$$x^2 - 3x + 2 = 2^2 - 3 \cdot 2 + 2 = 0$$

Multiple solutions are common for quadratic equations, so we will state answers using the roster notation.

ANSWER: $\{1, 2\}$

7.2. Equation Solving Strategy. We are already familiar with the procedure for solving *linear* polynomial equations, and now we can tackle all the rest of them. Faced with a polynomial equation, we will first try to determine whether it is linear or not. If it is linear, then we will solve it by isolating the variable. If it is a quadratic or a higher degree equation, then we will arrange it so that all the non-zero terms are on the same side, factor the polynomial completely, and take advantage of the zero product property.

EXAMPLE 7.2.1. Solve the equation

$$2x^2+x=1$$

SOLUTION: This equation appears to be quadratic, so we will add -1 to both sides in order to make the right side zero, then factor by guessing the coefficients.

$$\begin{aligned} 2x^2+x &= 1 \\ 2x^2+x-1 &= 0 \\ (2x-1)(x+1) &= 0 \end{aligned}$$

The left side is a product, and the right side is zero, so we can apply the zero product property. Solving the equation $2x-1=0$ gives us

$$\begin{aligned} 2x-1 &= 0 \\ 2x &= 1 \\ x &= 0.5 \end{aligned}$$

Solving the equation $x+1=0$ gives us $x=-1$.

ANSWER: $\{0.5,-1\}$

EXAMPLE 7.2.2. Solve the equation

$$8x^2-18=0$$

SOLUTION: All the non-zero terms of this quadratic equation are already on the left side, so let's try factoring it completely. Instead of factoring out 2 we opt to multiply both sides by 0.5, canceling it altogether.

$$\begin{aligned} 8x^2-18 &= 0 \\ 0.5(8x^2-18) &= 0.5(0) \\ 4x^2-9 &= 0 && \text{this is a difference of squares} \\ (2x)^2-3^2 &= 0 \\ (2x+3)(2x-3) &= 0 \end{aligned}$$

The left side is a product, and the right side is zero, so we can apply the zero product property. Solving the equation $2x+3=0$ gives us $x=-1.5$.

Solving the equation $2x-3=0$ gives us $x=1.5$.

ANSWER: $\{1.5, -1.5\}$

EXAMPLE 7.2.3. Solve the equation

$$14x^2 = -16x$$

SOLUTION: We will add $16x$ to both sides to make the right side zero, and try to factor. Notice that we opt to divide both sides of the equation by the common factor 2 rather than to factor it out, because dividing both sides by a non-zero number makes an equivalent equation.

$$\begin{aligned}
14x^2 &= -16x \\
14x^2+16x &= 0 && \text{divide both sides by 2} \\
\frac{14x^2+16x}{2} &= \frac{0}{2} \\
\frac{14x^2}{2}+\frac{16x}{2} &= 0 && \text{distributivity} \\
7x^2+8x &= 0 && \text{the GCF is } x \\
x(7x+8) &= 0 && \text{distributivity}
\end{aligned}$$

Notice we are not allowed to divide both sides of the equation by x. Unlike the non-zero quantity 2, x is an unknown quantity which may turn out to be zero.

The expression $x(7x+8)$ has two polynomial factors: x and $7x+8$. By the zero product property $x=0$ is a solution, and solving $7x+8=0$ gives us

$$\begin{aligned}
7x+8 &= 0 \\
7x &= -8 \\
x &= -8/7
\end{aligned}$$

ANSWER: $\{0, -8/7\}$

EXAMPLE 7.2.4. Solve the equation

$$4y^3 = 7y^2 + 15y$$

SOLUTION: This looks like a cubic equation, so we will subtract $(7y^2+15y)$ on both sides, and then factor completely.

$$\begin{aligned} 4y^3 &= 7y^2 + 15y \\ 4y^3 - 7y^2 - 15y &= 0 && \text{the GCF is } y \\ y(4y^2 - 7y - 15) &= 0 && \text{factor by guessing coefficients} \\ y(4y+5)(y-3) &= 0 \end{aligned}$$

By the zero product property, $y = 0$ is a solution;

solving $4y + 5 = 0$ yields $y = -1.25$;

and solving $y - 3 = 0$ yields $y = 3$.

As always the case with any equation, we can check the answers by substituting them and checking that the equation holds.

If $y = 0$ then

$$\begin{aligned} 4 \cdot 0^3 &= y \cdot 0^2 + 15 \cdot 0 \\ 0 &= 0 \end{aligned}$$

If $y = 3$ then

$$\begin{aligned} 4 \cdot 3^3 &= 7 \cdot 3^2 + 15 \cdot 3 \\ 108 &= 63 + 45 \\ 108 &= 108 \end{aligned}$$

Finally, if $y = -1.25$ then

$$\begin{aligned} 4(-1.25)^3 &= 7(-1.25)^2 + 15(-1.25) \\ -7.8125 &= 10.9375 - 18.75 \\ -7.8125 &= -7.8125 \end{aligned}$$

ANSWER: $\{-1.25, 0, 3\}$

Homework 6.7.

Solve the given equation by using the zero product property.

1. $(x+3)(x-10)=0$

2. $(x-2)(x+9)=0$

3. $(x-1)(x-8)=0$

4. $(x+4)(x+3)=0$

5. $3x(x+4)=0$

6. $2y(y-6)=0$

7. $(2x+3)(4x-5)=0$

8. $(5x-10)(6x+10)=0$

9. $x^2\left(x+\frac{1}{2}\right)=0$

10. $4y^3\left(y-\frac{2}{3}\right)=0$

11. $x(4x+0.8)(x-0.1)^2=0$

12. $y^2(3y-6)(y+1.7)=0$

Solve the given equation by factoring.

13. $x^2+7x+6=0$

14. $x^2+6x+5=0$

15. $y^2+4y-21=0$

16. $y^2=7y+18$

17. $x^2-11x+18=0$

18. $x^2-8x+15=0$

19. $y^2-6.5y=0$

20. $4z^2-12z=0$

21. $\frac{2}{3}x^2+\frac{1}{3}x=0$

22. $3z^2+4z=0$

23. $2x^2=72$

24. $6y^2-54=0$

25. $10x+x^2+25=0$

26. $6x+9+x^2=0$

27. $(x+1)(x-7)=-16$

28. $(x+2)(7-x)=18$

29. $(3x+5)(x+3)=7$

30. $(5x+4)(x-1)=2$

31. $81x^2-6=19$

32. $14x^2-3=42x-3$

Homework 6.7 Answers.

1. $\{-3, 10\}$

3. $\{1, 8\}$

5. $\{-4, 0\}$

7. $\left\{-\frac{3}{2}, \frac{5}{4}\right\}$

9. $\left\{-\frac{1}{2}, 0\right\}$

11. $\{-0.2, 0, 0.1\}$

13. $\{-6, -1\}$

15. $\{-7, 3\}$

17. $\{2, 9\}$

19. $\{0, 6.5\}$

21. $\left\{-\frac{1}{2}, 0\right\}$

23. $\{-6, 6\}$

25. $\{-5\}$

27. $\{3\}$

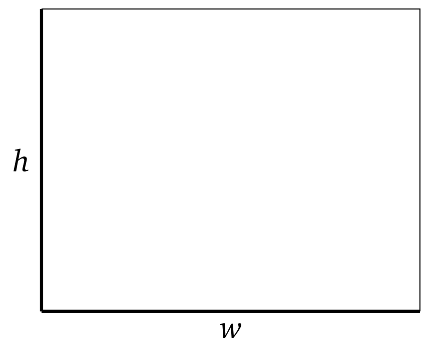
29. $\left\{-4, -\frac{2}{3}\right\}$

31. $\left\{-\frac{5}{9}, \frac{5}{9}\right\}$

8. Factoring Applications

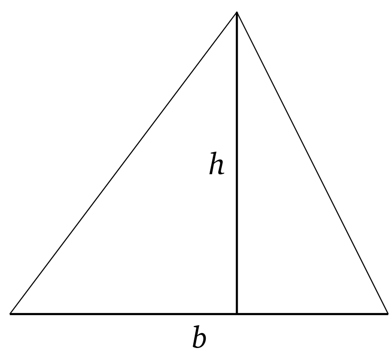
8.1. Applications to Areas. Recall that the area of a rectangle with width w and length l is the product of its dimensions.

$$A = w \cdot l$$



Recall also that the area of a triangle with base b and height h is one half of their product.

$$A = \frac{bh}{2}$$



EXAMPLE 8.1.1. The length of a rectangular carpet is 1 foot greater than the width. Find the dimensions of the carpet if its surface area is 12 square feet.

SOLUTION: Let w and l be the width and the length in feet, respectively. We can translate "the length is 1 foot greater than the width" as

$$l = w + 1$$

We can also write the equation for the area:

$$wl = 12$$

This is not a linear system, but we can solve it by substitution. The first equation is already solved for l, so we substitute $(w+1)$ for l in the area equation and solve for w:

$$\begin{aligned} wl &= 12 \\ w(w+1) &= 12 \\ w^2 + w &= 12 \end{aligned}$$

This is a quadratic equation, which we will solve by factoring.

$$\begin{aligned} w^2+w &= 12 && \text{put non-zero terms on one side} \\ w^2+w-12 &= 0 && \text{trinomial pattern } x^2+bx+c \\ (w-3)(w+4) &= 0 \end{aligned}$$

By the zero product property, the solutions are $w=3$ and $w=-4$. The width cannot be negative, so the solution -4 is not applicable. The only useful solution is 3. To find the length, we substitute 3 for w in the equation solved for l:

$$\begin{aligned} l &= w+1 \\ l &= (3)+1 \\ l &= 4 \end{aligned}$$

ANSWER:
w and l are width and length respectively, in feet

$$\begin{cases} l = w+1 \\ wl = 12 \end{cases}$$

solution: $w=3, \quad l=4$

EXAMPLE 8.1.2. The height of a triangular sail is 1 meter less than the twice the length of its base. Find the dimensions of the sail if its surface area is 14 square meters.

SOLUTION: Let b and h be the base and the height of the sail in meters. The sentence "the height of a triangular sail is 1 meter less than the twice the length of its base" can be translated as

$$h = 2b-1$$

We can also write the equation for the area:

$$bh/2 = 14$$

The first equation is already solved for h, so we can substitute $(2b-1)$ for h in the area equation and solve for b:

$$\begin{aligned} \frac{bh}{2} &= 14 && \text{area equation} \\ \frac{b(2b-1)}{2} &= 14 && \text{substitution} \\ b(2b-1) &= 28 && \text{multiplied both sides by 2} \end{aligned}$$

With the last step, we multiplied both sides by 2 to get rid of the fraction. The result looks like a quadratic equation, so we solve it by factoring:

$$\begin{aligned} 2b^2 - b &= 28 && \text{distributed on the left side} \\ 2b^2 - b - 28 &= 0 && \text{trinomial pattern } ax^2 + bx + c \\ (2b+7)(b-4) &= 0 && \text{factored by guessing coefficients} \end{aligned}$$

By the zero product property, either

$$\begin{aligned} 2b + 7 &= 0 \\ 2b &= -7 \\ b &= -7/2 \end{aligned}$$

or

$$\begin{aligned} b - 4 &= 0 \\ b &= 4 \end{aligned}$$

The negative solution $b = -7/2$ is not applicable because it cannot denote a length, so the only useful solution is $b = 4$. To find the height, substitute 4 for b in the equation solved for h:

$$\begin{aligned} h &= 2b - 1 \\ h &= 2(4) - 1 \\ h &= 7 \end{aligned}$$

ANSWER:
b and h are base and height respectively, in meters

$$\begin{cases} h = 2b - 1 \\ bh/2 = 14 \end{cases}$$

solution: $b = 4, \quad h = 7$

EXAMPLE 8.1.3. The height of a triangular tile is 40% greater than its base. Find the dimensions of the tile if its area is 17.5 square inches.

SOLUTION: Let b and h be the base and the height of the triangle in inches. The sentence "the height of a triangular tile is 40% greater than its base" is a statement about percent increase, and can be translated as

$$\begin{aligned} h &= b + 0.4b && \text{see definition 4.2.1 in chapter 2} \\ h &= 1.4b && \text{combined like terms} \end{aligned}$$

And the statement about the area can be translated as

$$\frac{bh}{2} = 17.5$$

Using the first equation, which is solved for h, we can substitute $1.4b$ for h into the second equation, and solve for b:

$$\begin{aligned}\frac{bh}{2} &= 17.5\\ \frac{b(1.4b)}{2} &= 17.5 && \text{substituted } 1.4b \text{ for } h\end{aligned}$$

To get rid of the fraction, we multiply both sides by 2:

$$\begin{aligned}b(1.4b) &= 35\\ 1.4b^2 &= 35 && \text{simplify left side}\\ b^2 &= 25 && \text{divide both sides by } 1.4\end{aligned}$$

This looks like a quadratic equation, so we solve it by factoring:

$$\begin{aligned}b^2 - 25 &= 0 && \text{subtracted 25 on both sides}\\ (b+5)(b-5) &= 0 && \text{difference of squares } b^2 - 5^2\end{aligned}$$

By the zero product property, the solutions are -5 and 5. Negative length does not make sense, so the negative solution is not applicable, and the only useful solution is $b = 5$. We can find the other dimension using the equation solved for h:

$$\begin{aligned}h &= 1.4b\\ h &= 1.4 \cdot 5\\ h &= 7\end{aligned}$$

ANSWER:

b and h are base and height respectively, in inches

$$\begin{cases} h = b + 0.4b \\ bh/2 = 17.5 \end{cases}$$

solution: $b = 5, \quad h = 7$

Homework 6.8.

1. The width of a rectangular rug is 4 times less than its length. Find the dimensions of the rug if its area is 36 square feet.

2. The length of a rectangular room is 6 yards greater than its width. Find the dimensions of the room if its area is 55 square yards.

3. The base of a triangle is 2 meters longer than twice its height. Find the base and the height if the area of the triangle is 12 square meters.

4. The base of a triangle is 2 units longer than the height. Find the base and the height if the area of the triangle is 12 square units.

5. The width of a rectangular laptop screen is 7 inches less than its length. Find the dimensions of the screen if its area is 144 square inches.

6. The width of a rectangular mural is 10% greater than its length. Find the dimensions of the mural if its area is 110 square meters.

7. Dmitriy buys a gold earring shaped like a thin flat triangle. Find the dimensions of the earring if the base of the triangle is 25% shorter than its height, and the area of the triangle is 24 mm^2.

8. The height of a triangular piece of fabric is 3 times greater than its base. Find the dimensions of the triangle if its area is 13.5 square inches.

Homework 6.8 Answers.

1.
w and l are width and length respectively, in feet

$$\begin{cases} w = l/4 \\ wl = 36 \end{cases}$$

solution: $w = 3, \quad l = 12$

3.
b and h are base and height respectively, in meters

$$\begin{cases} b = 2h + 2 \\ bh/2 = 12 \end{cases}$$

solution: $b = 8, \quad h = 3$

5.
w and l are width and length respectively, in inches

$$\begin{cases} w = l - 7 \\ wl = 144 \end{cases}$$

solution: $w = 9, \quad l = 16$

7.
b and h are base and height respectively, in mm

$$\begin{cases} b = h - 0.25h \\ bh/2 = 24 \end{cases}$$

solution: $b = 6, \quad h = 8$

Practice Test 6

Factor each of the following expressions completely:

1. $48x^7y^3 + 42x^4y^4$

2. $6x^4 - 42x^3 - 6x^2$

3. $x^2 - 10x + 24$

4. $18x^2 + 63x^2a - 24x - 84xa$

5. $5x^2 - 17x + 6$

6. $4y^2 + 10y - 6$

7. $1 - 16x^4$

Solve each of the following equations using the zero product property:

8. $x(x+4) = 0$

9. $10(x-6)(3x+1) = 0$

10. $-7x^2(x^2+1)(x-1) = 0$

Solve each of the following equations using factoring:

11. $x^2 + 100 = 20x$

12. $25x^2 - 9 = 0$

13. $3x^2 - 3x = 36$

14. $5x^3 = 45x$

15. $18x^2 - 3x = 6$

16. $x(2x-1) = 3$

17. $x^3 - 9x^2 + 2x - 18 = 0$

18. The length of a rectangular carpet is 5 times greater than the width, and the area of the carpet is 45 square feet. Find the dimensions of the carpet.

19. A piece of stained glass is shaped like a triangle, with its height 4 cm shorter than its base. Find the base and the height of the triangle if its area is 48 cm^2.

Practice Test 6 Answers.

1. $6x^4y^3(8x^3+7y)$

2. $6x^2(x^2-7x-1)$

3. $(x-4)(x-6)$

4. $3x(3x-4)(2+7a)$

5. $(5x-2)(x-3)$

6. $2(2y-1)(y+3)$

7. $(1+4x^2)(1+2x)(1-2x)$

8. $\{0,-4\}$

9. $\{6,-1/3\}$

10. $\{0,1\}$

11. $\{10\}$

12. $\{-3/5,3/5\}$

13. $\{4,-3\}$

14. $\{0,3,-3\}$

15. $\{2/3,-1/2\}$

16. $\{3/2,-1\}$

17. $\{9\}$

18.
l and w are length and width respectively, in feet

$$\begin{cases} lw = 45 \\ l = 5w \end{cases}$$

solution: $l = 15, \quad w = 3$

19.
b and h are base and height respectively, in cm

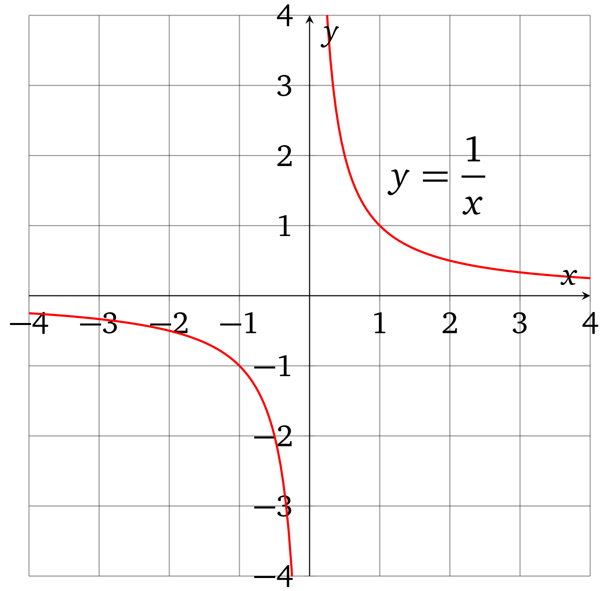
$$\begin{cases} bh/2 = 48 \\ h = b-4 \end{cases}$$

solution: $b = 12, \quad h = 8$

CHAPTER 7

Rational Expressions

1. Simplifying Rational Expressions



1.1. Definition.

DEFINITION 1.1.1. A *rational expression* is either a polynomial expression or a quotient of a polynomial and a non-zero polynomial.

BASIC EXAMPLE 1.1.1. Here are some rational expressions:

$$x^2 - x \qquad \frac{5-x}{4} \qquad \frac{1}{x} \qquad \frac{1+x^2}{x-2} \qquad \frac{x^2+2x+1}{x+1} \qquad \frac{x+yw}{xy-w^5}$$

On the other hand

$$\frac{1}{x} + \frac{1}{x}$$

is not a rational expression, since it is neither a polynomial nor a quotient of polynomials.

1.2. Variable Restrictions. Unlike polynomials, which can always be evaluated, rational expressions will occasionally fail to have a defined value. As the most basic example, the expression $1/x$ can be evaluated for every real number x except for $x = 0$. Because of that, the process of simplifying rational expressions may well produce expressions which are not equivalent, but *almost equivalent*, with only a finite number of disagreements.

BASIC EXAMPLE 1.2.1. Expressions

$$\frac{(x+2)(x-1)}{x-1} \qquad \text{and} \qquad x+2$$

are not equivalent even though they agree on the value for *almost* all real numbers x. But when $x = 1$,

$$x+2 = 1+2 = 3$$

whereas

$$\frac{(x+2)(x-1)}{x-1} = \frac{(1+2)(1-1)}{1-1} = \frac{3\cdot 0}{0}$$

which is undefined.

In this text we will simplify rational expressions by canceling common polynomial factors, even though in many cases it will result in non-equivalent expressions. Our excuse will be that we can find exactly which inputs lead to disagreements.

EXAMPLE 1.2.1. Find all variable values which make the given expression undefined:

$$\frac{x+1}{x-5}$$

SOLUTION: A rational expression is a quotient of two polynomials. The numerator is always defined, and the denominator is always defined. The only operation which may possibly fail is the division: the fraction bar. It will fail just in case if the denominator is equal to zero. So we can make the denominator equal to zero, and then solve for x. We ignore the numerator completely.

$$\begin{aligned} x-5 &= 0 \\ x &= 5 \end{aligned}$$

The only number which makes this expression undefined is $x = 5$.

ANSWER: $\{5\}$

EXAMPLE 1.2.2. Find all variable values which make the given expression undefined:

$$\frac{x+4}{x^2-16}$$

SOLUTION: We can make the denominator equal to zero, and then solve for x. We ignore the numerator completely, even though it happens to have the factor $(x+4)$, which can also be found in the denominator.

$$\begin{aligned} x^2-16 &= 0 \\ (x+4)(x-4) &= 0 \end{aligned}$$

By the zero product property, x is either -4 or 4.

ANSWER: $\{-4,4\}$

EXAMPLE 1.2.3. Find all variable values which make the given expression undefined:

$$\frac{\frac{1}{x}+\frac{1}{7}}{\frac{x}{6}-\frac{2}{9}}$$

SOLUTION: There are two variable denominators in this expression, and if either one happens to be 0, then the whole expression is undefined. One variable denominator is x from $1/x$ up on top, so $x=0$ will make the expression undefined. The other variable denominator is

$$\frac{x}{6}-\frac{2}{9}$$

so we make it equal to zero and solve for x:

$$\begin{aligned} \frac{x}{6}-\frac{2}{9} &= 0 \\ \frac{x}{6} &= \frac{2}{9} \\ x &= \frac{4}{3} \end{aligned}$$

ANSWER: $\{0,4/3\}$

1.3. Canceling Common Polynomial Factors. We can simplify rational expressions by fully factoring both numerator and denominator, and then canceling common polynomial factors. In answers, we will write simplified rational expressions as quotients of polynomial factorizations. This way we can be sure nothing else can be canceled.

EXAMPLE 1.3.1. Simplify and state the answer as a polynomial:

$$\frac{a^2-1}{a+1}$$

SOLUTION:

$$\begin{aligned}\frac{a^2-1}{a+1} &= \frac{(a+1)(a-1)}{(a+1)} \\ &= a-1\end{aligned}$$

ANSWER: $a-1$

EXAMPLE 1.3.2. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{6x-3}{4x^2-4x+1}$$

SOLUTION:

$$\begin{aligned}\frac{6x-3}{4x^2-4x+1} &= \frac{3(2x-1)}{(2x-1)(2x-1)} \\ &= \frac{3}{2x-1}\end{aligned}$$

ANSWER: $\frac{3}{2x-1}$

EXAMPLE 1.3.3. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{4a^2-5ab}{30b-24a}$$

SOLUTION:

$$\frac{4a^2-5ab}{30b-24a} = \frac{a(4a-5b)}{6(5b-4a)}$$

The factors $(4a-5b)$ and $(5b-4a)$ are not identical, so they cannot be canceled outright. But they are opposites of each other, so the quotient is equivalent to -1 whenever defined. Algebraically, we can make them look exactly the same if we factor out -1.

$$\begin{aligned}\frac{a(4a-5b)}{6(5b-4a)} &= \frac{a(-1)(-4a+5b)}{6(5b-4a)} \\ &= \frac{a(-1)(5b-4a)}{6(5b-4a)} && \text{now they are identical and can be canceled} \\ &= \frac{-a}{6}\end{aligned}$$

ANSWER: $\dfrac{-a}{6}$

EXAMPLE 1.3.4. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{x^3+2x^2+x+2}{x^4-1}$$

SOLUTION: The numerator has 4 terms, so we will attempt grouping; the denominator is a difference of squares. We factor polynomials completely before rewriting the rational expression.

$$\begin{aligned}x^3+2x^2+x+2 &= x^2(x+2)+1(x+2) \\ &= (x^2+1)(x+2)\end{aligned}$$

x^2+1 is an irreducible quadratic, and $x+2$ is linear, so the numerator is fully factored.

$$\begin{aligned}x^4-1 &= (x^2+1)(x^2-1) && \text{difference of squares } \left(x^2\right)^2-1^2 \\ &= (x^2+1)(x+1)(x-1) && \text{difference of squares } x^2-1^2\end{aligned}$$

x^2+1 is an irreducible quadratic, and the other two factors are linear, so the denominator is fully factored. Now we can rewrite the rational expression in a fully factored form and cancel common polynomial factors:

$$\begin{aligned}\frac{x^3+2x^2+x+2}{x^4-1} &= \frac{(x^2+1)(x+2)}{(x^2+1)(x+1)(x-1)} \\ &= \frac{(x+2)}{(x+1)(x-1)}\end{aligned}$$

ANSWER: $\dfrac{x+2}{(x+1)(x-1)}$

Homework 7.1.

Find all variable values which make the given expression undefined.

1. $\dfrac{3k^2+30k}{k+10}$

2. $\dfrac{15n^2}{10n+25}$

3. $\dfrac{27p}{18p^2-36p}$

4. $\dfrac{x+10}{8x^2+80x}$

5. $\dfrac{10m^2+8m}{10m}$

6. $\dfrac{10x+16}{6x+20}$

7. $\dfrac{b^2+12b+32}{b^2-4b-32}$

8. $\dfrac{10y^2+30y}{35y^2-5y}$

9. $\dfrac{w^2-1}{w^2+1}$

10. $\dfrac{16x^2+25}{16x^2-25}$

Simplify the given expression and state the answer as a quotient of polynomial factorizations.

11. $\dfrac{21x^2}{18x}$

12. $\dfrac{12n}{4n^2}$

13. $\dfrac{24a}{40a^2}$

14. $\dfrac{21k}{24k^3}$

15. $\dfrac{32x^3y}{8xy^2}$

16. $\dfrac{90x^2}{20x^2y}$

17. $\dfrac{18m-24}{60}$

18. $\dfrac{20}{4p+2}$

19. $\dfrac{x+1}{x^2+8x+7}$

20. $\dfrac{32x^2}{28x^2+28}$

21. $\dfrac{n^2+4n-12}{n^2-7n+10}$

22. $\dfrac{b^2+14b+48}{b^2+15b+56}$

23. $\dfrac{9v+54}{v^2-4v-60}$

24. $\dfrac{30x-90}{50x+40}$

25. $\dfrac{12x^2-42x}{30x^2-42x}$

26. $\dfrac{6a-10a^2}{10a+4a^2}$

27. $\dfrac{k^2-12k+32}{k^2-64}$

28. $\dfrac{9p+18}{p^2+4p+4}$

29. $\dfrac{9n^2+89n-10}{9n+90}$

30. $\dfrac{3x^2-29x+40}{5x^2-30x-80}$

31. $\dfrac{2x^2-10x+8}{3x^2-7x+4}$

32. $\dfrac{7n^2-32n+16}{4n-16}$

33. $\dfrac{n^2-2n+1}{6n+6}$

34. $\dfrac{7a^2-26a-45}{6a^2-34a+20}$

35. $\dfrac{56x-48}{24x^2+56x+32}$

36. $\dfrac{4k^3-2k^2-2k}{9k^3-18k^2+9k}$

Homework 7.1 Answers.

1. $\{-10\}$

3. $\{0,2\}$

5. $\{0\}$

7. $\{-4,8\}$

9. $\varnothing$

11. $\dfrac{7x}{6}$

13. $\dfrac{3}{5a}$

15. $\dfrac{4x^2}{y}$

17. $\dfrac{3m-4}{10}$

19. $\dfrac{1}{x+7}$

21. $\dfrac{n+6}{n-5}$

23. $\dfrac{9}{v-10}$

25. $\dfrac{2x-7}{5x-7}$

27. $\dfrac{k-4}{k+8}$

29. $\dfrac{9n-1}{9}$

31. $\dfrac{2(x-4)}{3x-4}$

33. $\dfrac{(n-1)^2}{6(n+1)}$

35. $\dfrac{7x-6}{(3x+4)(x+1)}$

2. Products of Rational Expressions

2.1. Multiplying Rational Expressions. We can multiply rational expressions just like we would multiply fractions: the numerator of the product is the product of numerators, and the denominator of the product is the product of denominators. Rather than to carry out the multiplication of polynomials, we will factor them, so that we can cancel common polynomial factors and leave the answer in a simplified form.

EXAMPLE 2.1.1. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{25x^2}{9y^8} \cdot \frac{24y^4}{55x^7}$$

SOLUTION: We begin by multiplying across and then we cancel common integer and variable factors.

$$\begin{aligned}
\frac{25x^2}{9y^8} \cdot \frac{24y^4}{55x^7} &= \frac{25 \cdot 24 \cdot x^2 y^4}{9 \cdot 55 \cdot y^8 x^7} \\
&= \frac{(5 \cdot 5)(2 \cdot 2 \cdot 2 \cdot 3)}{(3 \cdot 3)(5 \cdot 11) \cdot x^5 y^4} && \text{cancel variables, factor integers} \\
&= \frac{5 \cdot 2 \cdot 2 \cdot 2}{3 \cdot 11 \cdot x^5 y^4} && \text{canceled common prime factors} \\
&= \frac{40}{33x^5 y^4}
\end{aligned}$$

ANSWER: $\frac{40}{33x^5 y^4}$

EXAMPLE 2.1.2. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{x^2-4}{x+3} \cdot \frac{x+1}{x^2+3x+2}$$

SOLUTION: We can multiply across, then factor all the polynomials, and cancel the common polynomial factors. Note that the invisible parentheses of the fraction notation become visible once we rewrite the expression as a single fraction.

$$\frac{x^2-4}{x+3} \cdot \frac{x+1}{x^2+3x+2} = \frac{(x^2-4)(x+1)}{(x+3)(x^2+3x+2)}$$

The difference of squares $x^2 - 4$ factors as

$$x^2 - 4 = (x+2)(x-2)$$

while the trinomial $x^2 + 3x + 2$ factors as

$$x^2 + 3x + 2 = (x+1)(x+2)$$

Now we can rewrite the original fraction in a fully factored form, and cancel common polynomial factors.

$$\begin{aligned} \frac{(x^2-4)(x+1)}{(x+3)(x^2+3x+2)} &= \frac{(x+2)(x-2)(x+1)}{(x+3)(x+1)(x+2)} \\ &= \frac{(x-2)}{(x+3)} && \text{common polynomial factors canceled} \end{aligned}$$

ANSWER: $\dfrac{x-2}{x+3}$

2.2. Dividing Rational Expressions.

EXAMPLE 2.2.1. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{14}{25} \div \frac{21}{10}$$

SOLUTION: Recall that division amounts to multiplying by the reciprocal of the divisor:

$$\frac{14}{25} \div \frac{21}{10} = \frac{14}{25} \cdot \frac{10}{21} = \frac{14 \cdot 10}{25 \cdot 21}$$

Instead of finding these large products, we opt to factor the integers completely so that it is easier to cancel all the common factors.

$$\begin{aligned} \frac{14 \cdot 10}{25 \cdot 21} &= \frac{(2 \cdot 7)(2 \cdot 5)}{(5 \cdot 5)(3 \cdot 7)} \\ &= \frac{2 \cdot 2}{5 \cdot 3} \\ &= \frac{4}{15} \end{aligned}$$

ANSWER: 4/15

EXAMPLE 2.2.2. Simplify and state the answer as a polynomial or a quotient of polynomial factorizations:

$$\frac{x^2+8x+16}{x} \div \frac{x+4}{x^2}$$

SOLUTION: First we rewrite the division as a multiplication by the reciprocal, and then multiply fractions as usual:

$$\begin{aligned} \frac{x^2+8x+16}{x} \div \frac{x+4}{x^2} &= \frac{x^2+8x+16}{x} \cdot \frac{x^2}{x+4} && \text{when we multiply sums} \\ &= \frac{(x^2+8x+16)x^2}{x(x+4)} && \text{fraction parentheses become visible} \end{aligned}$$

Now we can factor the polynomials completely and cancel all common factors. $x^2+8x+16$ is a square of a binomial, and factors as $(x+4)^2$, so we can write

$$\begin{aligned} \frac{(x^2+8x+16)x^2}{x(x+4)} &= \frac{(x+4)^2x^2}{x(x+4)} \\ &= (x+4)x && \text{common factors } x \text{ and } (x+4) \text{ cancel} \end{aligned}$$

For the sake of consistency, we will state polynomial answers in the standard form.

ANSWER: x^2+4x

EXAMPLE 2.2.3. Simplify and state the answer as a polynomial or a quotient of polynomial factorizations:

$$\frac{3x^2-75}{5+x} \div \frac{5-x}{12}$$

SOLUTION: First we rewrite the division as a multiplication by the reciprocal, and then multiply fractions as usual:

$$\begin{aligned} \frac{3x^2-75}{5+x} \div \frac{5-x}{12} &= \frac{3x^2-75}{5+x} \cdot \frac{12}{5-x} \\ &= \frac{(3x^2-75)12}{(5+x)(5-x)} \end{aligned}$$

$3x^2-75$ has GCF 3 and then factors as a difference of squares:

$$\begin{aligned} 3x^2-75 &= 3(x^2-25) \\ &= 3(x^2-5^2) \\ &= 3(x+5)(x-5) \end{aligned}$$

So in a fully factored form, the rational expression looks like this:

$$\begin{aligned}\frac{(3x^2-75)12}{(5+x)(5-x)} &= \frac{3(x+5)(x-5)12}{(5+x)(5-x)} \\ &= \frac{36(x-5)}{5-x} && \text{common factor } (x+5) \text{ canceled}\end{aligned}$$

Note that we canceled $x+5$ with $5+x$ because they are equivalent, but we can not cancel $x-5$ with $5-x$ as easily because they are opposites, and dividing a number by its opposite results in -1.

$$\begin{aligned}\frac{36(x-5)}{5-x} &= \frac{36(x-5)}{(-1)(-5+x)} \\ &= \frac{36(x-5)}{(-1)(x-5)} \\ &= \frac{36}{-1} && \text{common factor } (x-5) \text{ canceled} \\ &= -36\end{aligned}$$

ANSWER: -36

Homework 7.2.

Simplify and state the answer as a polynomial or a quotient of polynomial factorizations.

1. $\dfrac{8x^2}{9} \cdot \dfrac{9}{2}$

2. $\dfrac{20n}{3n} \cdot \dfrac{7}{5n}$

3. $\dfrac{5x^2}{4} \div \dfrac{5}{6}$

4. $\dfrac{10p}{5} \div \dfrac{8}{10}$

5. $\dfrac{7(m-6)}{m-6} \cdot \dfrac{5m(7m-5)}{7(7m-5)}$

6. $\dfrac{7r}{7r(r+10)} \cdot \dfrac{(r-6)^2}{r-6}$

7. $\dfrac{7}{10(n+3)} \div \dfrac{n-2}{(n+3)(n-2)}$

8. $\dfrac{6x(x+4)}{x-3} \div \dfrac{6x(x-6)}{(x-3)(x-6)}$

9. $\dfrac{25n+25}{5} \cdot \dfrac{4}{30n+30}$

10. $\dfrac{v-1}{4} \cdot \dfrac{4}{v^2-11v+10}$

11. $\dfrac{9}{b^2-b-12} \div \dfrac{b-5}{b^2-b-12}$

12. $\dfrac{x-10}{35x+21} \div \dfrac{7}{35x+21}$

13. $\dfrac{x^2-6x-7}{x+5} \cdot \dfrac{x+5}{x-7}$

14. $\dfrac{1}{a-6} \cdot \dfrac{8a+80}{8}$

15. $\dfrac{8k}{24k^2-40k} \div \dfrac{1}{15k-25}$

16. $\dfrac{p-8}{p^2-12p+32} \div \dfrac{1}{p-10}$

17. $(n-8) \cdot \dfrac{6}{10n-80}$

18. $\dfrac{3x-6}{12x-24} \cdot (x+3)$

19. $\dfrac{x^2-7x+10}{x-2} \div \dfrac{x^2-x-20}{x+10}$

20. $\dfrac{21v^2+16v-16}{3v+4} \div \dfrac{35v-20}{v-9}$

21. $\dfrac{b+2}{40b^2-24b} \cdot (5b-3)$

22. $\dfrac{2n^2-12n-54}{n+7} \cdot \dfrac{1}{2n+6}$

23. $\dfrac{x^2+11x+24}{6x^3+18x^2} \div \dfrac{x^2+5x-24}{6x^3+6x^2}$

24. $\dfrac{n-7}{6n-12} \div \dfrac{n^2-13n+42}{12-6n}$

25. $\dfrac{k-7}{k^2-k-12} \cdot \dfrac{7k^2-28k}{8k^2-56k}$

26. $\dfrac{x^2-12x+32}{x^2-6x-16} \cdot \dfrac{7x^2+14x}{7x^2+21x}$

27. $\dfrac{n-7}{n^2-2n-35} \div \dfrac{9n+54}{10n+50}$

28. $\dfrac{27a+36}{9a+63} \div \dfrac{6a+8}{2}$

29. $\dfrac{x^2-1}{2x-4} \cdot \dfrac{x^2-4}{x^2-x-2} \div \dfrac{x^2+x-2}{3x-6}$

30. $\dfrac{x^2+3x-10}{x^2+6x+5} \div \dfrac{8x+20}{6x+15} \cdot \dfrac{2x^2-x-3}{2x^2+x-6}$

Homework 7.2 Answers.

1. $4x^2$

3. $\frac{3x^2}{2}$

5. $5m$

7. $\frac{7}{10}$

9. $\frac{2}{3}$

11. $\frac{9}{b-5}$

13. $x+1$

15. 5

17. $\frac{3}{5}$

19. $\frac{x+10}{x+4}$

21. $\frac{b+2}{8}$

23. $\frac{x+1}{x-3}$

25. $\frac{7}{8(k+3)}$

27. $\frac{10}{9(n+6)}$

29. $\frac{3}{2}$

3. LCD for Rational Expressions

3.1. LCM for Polynomials.

DEFINITION 3.1.1. A *common multiple* for a collection of polynomials P, Q, R, ... with integer coefficients is a polynomial which is divisible by P, Q, R, ... without a remainder, using the polynomial long division.

The *least common multiple* (or *LCM*) for a collection of polynomials with integer coefficients is a common multiple of the least possible polynomial degree and with coefficients as low as possible.

The *lowest common denominator* (or *LCD*) for a collection of rational expressions is the LCM of their denominators.

THEOREM 3.1.1. To find the least common multiple for a collection of polynomials with integer coefficients, we can factor each polynomial completely, and then take the product with each prime factor and each irreducible polynomial factor to the highest degree we found. Note that the integer factor of the polynomial LCM is the LCM for the collection of individual integer factors.

EXAMPLE 3.1.1. Find the LCM for given polynomial expressions and state it in a fully factored form.

$$x^4, \quad x, \quad x^7$$

SOLUTION: These polynomials are already fully factored. The only factor here is x, and its highest degree is 7, so LCM is x^7.

ANSWER:

EXAMPLE 3.1.2. Find the LCM for given polynomial expressions and state it in a fully factored form.

$$4m^2, \quad 6mn, \quad 2n^2$$

SOLUTION: These polynomials are already fully factored. The LCM for integer factors 4, 6, and 2 is 12, the highest degree of m is 2, and the highest degree of n is also 2. So the LCM is $12m^2n^2$.

ANSWER: $12m^2n^2$

EXAMPLE 3.1.3. Find the LCM for given polynomial expressions and state it in a fully factored form.

$$25x^2, \quad 5x^2-5, \quad 10x^2-20x+10$$

SOLUTION: We factor the polynomials first:

$$\begin{aligned} & 25x^2 & \text{already factored} \\ 5x^2-5 &= 5(x^2-1) & \text{GCF} \\ &= 5(x+1)(x-1) & \text{difference of squares} \\ 10x^2-20x+10 &= 10(x^2-2x+1) & \text{GCF} \\ &= 10(x-1)^2 & \text{special product pattern} \end{aligned}$$

The LCM for the integer factors 5, 25, and 10 is 50. The highest degree of $(x-1)$ is 2, the highest degree of x is 2, and the highest degree of $(x+1)$ is 1. Hence the polynomial LCM is $50x^2(x+1)(x-1)^2$.

ANSWER: $50x^2(x+1)(x-1)^2$

3.2. LCD for Rational Expressions.

EXAMPLE 3.2.1. Find the LCD for rational expressions

$$\frac{1}{2x}, \quad \frac{1}{x^3}$$

SOLUTION: These denominators are already fully factored. The highest degree of the factor 2 is 1, and the highest degree of the factor x is 3, so LCD $= 2^1 \cdot x^3$.

ANSWER: $2x^3$

EXAMPLE 3.2.2. Find the LCD for rational expressions

$$\frac{1}{25x^2y}, \quad \frac{1}{20x^3y^2}, \quad \frac{1}{2xw^7}$$

SOLUTION: In the examples above we found the integer factor for the polynomial LCM by taking the LCM of individual integer factors. In more complicated situations we may have to find integer factorizations in order to compute the LCM.

We factor the denominators first:

$$\begin{aligned} 25x^2y &= 5^2 \cdot x^2y \\ 20x^3y^2 &= 2^2 \cdot 5 \cdot x^3y^2 \\ & 2xw^7 && \text{already factored} \end{aligned}$$

The highest degree of 2 is 2, the highest degree of 5 is 2, the highest degree of x is 3, the highest degree of y is 2, and the highest degree of w is 7, so LCD $= 2^2 \cdot 5^2 \cdot x^3y^2w^7$

ANSWER: $100x^3y^2w^7$

EXAMPLE 3.2.3. Find the LCD for rational expressions

$$\frac{x-5}{4x-4}, \quad \frac{7}{6x^2-6}$$

SOLUTION: Recall that for the purpose of finding the LCD, we are free to ignore the numerators completely. We factor the denominators first:

$$\begin{aligned} 4x-4 &= 2^2(x-1) \\ 6x^2-6 &= 2 \cdot 3(x^2-1) \\ &= 6(x+1)(x-1) \end{aligned}$$

The highest degree of 2 is 2, the highest degree of 3 is 1, the highest degree of $(x-1)$ is 1, and the highest degree of $(x+1)$ is 1, so LCD $= 2^2 \cdot 3 \cdot (x+1)(x-1)$.

ANSWER: $12(x+1)(x-1)$

EXAMPLE 3.2.4. Find the LCD for rational expressions

$$\frac{x^4}{x^2-x-6}, \quad \frac{19x}{x^2y-6xy+9y}$$

SOLUTION: Recall that for the purpose of finding the LCD, we are free to ignore the numerators completely. We factor the denominators first:

$$\begin{aligned} x^2-x-6 &= (x-3)(x+2) \\ x^2y-6xy+9y &= y(x^2-6x+9) && \text{GCF} \\ &= y(x-3)(x-3) && \text{special product pattern} \\ &= y(x-3)^2 \end{aligned}$$

The highest degree of y is 1, the highest degree of $(x-3)$ is 2, and the highest degree of $(x+2)$ is 1, so LCD $= y(x-3)^2(x+2)$

ANSWER: $y(x-3)^2(x+2)$

EXAMPLE 3.2.5. Find the LCD for rational expressions

$$\frac{1}{x^2-6x+9}, \quad \frac{1}{3x-x^2}$$

SOLUTION: We factor the denominators first:

$$\begin{aligned} x^2-6x+9 &= (x-3)^2 \\ 3x-x^2 &= x(3-x) \end{aligned}$$

The highest degree of x is 1, and the highest degree of $(x-3)$ is 2. $(3-x)$ is not equivalent to $(3-x)$, but we can make it equivalent by factoring out -1. In the interest of keeping the overall degree of the LCD as low as possible, we treat the factor $(3-x)$ as if it was $(x-3)$ with degree 1. Hence the LCD is $x(x-3)^2$.

ANSWER: $x(x-3)^2$

Homework 7.3.

Find the least common multiple for each collection of polynomial expressions and state it in a fully factored form:

1. $2a^3, \quad 6a^4b^2, \quad 4a^3b^5$

2. $5x^2y, \quad 25x^3y^5z$

3. $x^2-3x, \quad x-3, \quad x$

4. $4x-8, \quad x-2, \quad 4$

5. $x+2, \quad x-4$

6. $x, \quad x-7, \quad x+1$

7. $x^2-25, \quad x+5$

8. $x^2-9, \quad x^2-6x+9$

9. $x^2+3x+2, \quad x^2+5x+6$

10. $x^2-7x+10, \quad x^2-2x-15, \quad x^2+x-6$

Find the least common denominator for each collection of rational expressions and state it in a fully factored form:

11. $\dfrac{1}{a+1}, \quad \dfrac{1}{(a-1)^2}, \quad \dfrac{1}{a^2-1}$

12. $\dfrac{1}{x-2}, \quad \dfrac{1}{(x+2)^2}, \quad \dfrac{1}{x^2-4}$

13. $\dfrac{2n-1}{2n^2+n-1}, \quad \dfrac{n+1}{2n^2+3n-2}$

14. $\dfrac{4}{m^2-2m-3}, \quad \dfrac{m}{2m^2+3m+1}$

15. $\dfrac{1}{t-3}, \quad \dfrac{2}{t+3}, \quad \dfrac{3}{t^2-9}$

16. $\dfrac{5}{a-5}, \quad \dfrac{10}{a^2-10a+25}$

17. $\dfrac{5x}{x^2-9}, \quad \dfrac{2x}{x^2+11x+24}$

18. $\dfrac{2x}{x^2-4}, \quad \dfrac{4x}{x^2+5x+6}$

19. $\dfrac{x+3}{6x^3-24x^2+18x}, \quad \dfrac{x-1}{4x^5-24x^4+20x^3}$

20. $\dfrac{1}{9y^3-9y^2-18y}, \quad \dfrac{1}{6y^5-24y^4+24y^3}$

Homework 7.3 Answers.

1. $12a^4b^5$

3. $x(x-3)$

5. $(x+2)(x-4)$

7. $(x+5)(x-5)$

9. $(x+1)(x+2)(x+3)$

11. $(a+1)(a-1)^2$

13. $(2n-1)(n+1)(n+2)$

15. $(t+3)(t-3)$

17. $(x+3)(x-3)(x+8)$

19. $12x^3(x-1)(x-3)(x-5)$

4. Sums of Rational Expressions

4.1. Sums with a Common Denominator. When simplifying a sum with a common denominator, we take the sum of numerators over the common denominator, and then simplify the resulting rational expression.

EXAMPLE 4.1.1. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{4x}{x^2-1}+\frac{4}{x^2-1}$$

SOLUTION: The denominators are the same, so we can add the numerators over the common denominator, then factor the numerator, and cancel common polynomial factors, if any.

$$\begin{aligned}\frac{4x}{x^2-1}+\frac{4}{x^2-1} &= \frac{4x+4}{x^2-1}\\ &= \frac{4(x+1)}{(x+1)(x-1)}\\ &= \frac{4}{x-1}\end{aligned}$$

ANSWER: $\dfrac{4}{x-1}$

EXAMPLE 4.1.2. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{4x-1}{x+4}-\frac{2x-9}{x+4}$$

SOLUTION: The denominators are the same, so we can subtract the numerators over the common denominator, combine like terms, factor the numerator, and cancel any common polynomial factors we find. Notice that we are subtructing a sum, so we have to make the fraction parentheses visible.

$$\begin{aligned}\frac{4x-1}{x+4}-\frac{2x-9}{x+4} &= \frac{4x-1-(2x-9)}{x+4} && \text{subtract the entire numerator}\\ &= \frac{4x-1-2x+9}{x+4} && \text{add the sum of opposites}\\ &= \frac{2x+8}{x+4} && \text{combine like terms}\end{aligned}$$

Now we will factor polynomials completely and cancel identical polynomial factors:

$$\begin{aligned}\frac{2x+8}{x+4} &= \frac{2(x+4)}{x+4} \\ &= 2\end{aligned}$$

ANSWER: 2

4.2. General Sums. The most economical way to add rational expressions with different denominators is by rewriting them all with the LCD.

EXAMPLE 4.2.1. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{3}{10x} - \frac{4}{15x^2}$$

SOLUTION: LCD $= 30x^2$, so before we can subtract, we need to modify each fraction to have this denominator.

$$\begin{aligned}\frac{3}{10x} - \frac{4}{15x^2} &= \frac{3}{10x} \cdot \frac{3x}{3x} - \frac{4}{15x^2} \cdot \frac{2}{2} \\ &= \frac{9x}{30x^2} - \frac{8}{30x^2}\end{aligned}$$

Now that the denominators are the same, we can subtract the numerators:

$$\frac{9x}{30x^2} - \frac{8}{30x^2} = \frac{9x-8}{30x^2}$$

Both the numerator and the denominator are fully factored, so we are done.

ANSWER: $\dfrac{9x-8}{30x^2}$

EXAMPLE 4.2.2. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{x}{x-1}+\frac{1}{x}$$

SOLUTION: LCD $= x(x-1)$, so before we can add, we need to modify each fraction to have this denominator.

$$\begin{aligned} \frac{x}{x-1}+\frac{1}{x} &= \frac{x}{x-1}\cdot\frac{x}{x}+\frac{1}{x}\cdot\frac{x-1}{x-1} && \text{when multiplying sums} \\ &= \frac{x^2}{x(x-1)}+\frac{x-1}{x(x-1)} && \text{show fraction parentheses} \end{aligned}$$

Now that the denominators are the same, we can add the numerators:

$$\begin{aligned} \frac{x^2}{x(x-1)}+\frac{x-1}{x(x-1)} &= \frac{x^2+(x-1)}{x(x-1)} \\ &= \frac{x^2+x-1}{x(x-1)} \end{aligned}$$

The polynomial in the numerator is prime, so everything is fully factored, and we are done.

ANSWER: $\frac{x^2+x-1}{x(x-1)}$

EXAMPLE 4.2.3. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{2}{x^2+3x+2}-\frac{x}{x^2+5x+6}$$

SOLUTION: To find the LCD, we factor the denominators:

$$\begin{aligned} x^2+3x+2 &= (x+1)(x+2) \\ x^2+5x+6 &= (x+2)(x+3) \end{aligned}$$

So LCD $=(x+1)(x+2)(x+3)$, and before we can subtract fractions, we need to rewrite them with this denominator. To do so, we need to multiply the first fraction by

$$\frac{x+3}{x+3}$$

and the second fraction by

$$\frac{x+1}{x+1}$$

The result looks like this:

$$\begin{aligned}\frac{2}{x^2+3x+2}-\frac{x}{x^2+5x+6} &= \frac{2}{(x+1)(x+2)}-\frac{x}{(x+2)(x+3)}\\ &= \frac{2}{(x+1)(x+2)}\cdot\frac{x+3}{x+3}-\frac{x}{(x+2)(x+3)}\cdot\frac{x+1}{x+1}\\ &= \frac{2(x+3)}{(x+1)(x+2)(x+3)}-\frac{x(x+1)}{(x+2)(x+3)(x+1)}\end{aligned}$$

Now that the denominators are equivalent, we can subtract the numerators:

$$= \frac{2(x+3)-x(x+1)}{(x+1)(x+2)(x+3)}$$

Finally, we will simplify the numerator by combining like terms, factoring it, and then canceling common polynomial factors, if any:

$$\begin{aligned}&= \frac{2x+6-x^2-x}{(x+1)(x+2)(x+3)}\\ &= \frac{-x^2+x+6}{(x+1)(x+2)(x+3)}\end{aligned}$$

If we factor out -1 in the numerator, then we can detect an easy trinomial pattern and factor it as a product of two binomials:

$$\begin{aligned}&= \frac{-(x^2-x-6)}{(x+1)(x+2)(x+3)}\\ &= \frac{-(x-3)(x+2)}{(x+1)(x+2)(x+3)}\\ &= \frac{-(x-3)}{(x+1)(x+3)}\end{aligned}$$

ANSWER: $\dfrac{-x+3}{(x+1)(x+3)}$

EXAMPLE 4.2.4. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{4a}{a-2}+\frac{7a}{2-a}$$

SOLUTION: Since $a-2$ and $2-a$ are opposites of each other, we treat them as the same polynomial factor, and the LCD is $(a-2)$. Before we can add, though, we need to factor out -1 in one of the denominators, so that they become equivalent.

$$\frac{4a}{a-2}+\frac{7a}{2-a} \quad = \quad \frac{4a}{a-2}+\frac{7a}{-1(a-2)}$$

Because -1 is its own reciprocal, we can move the factor -1 into the numerator, making the denominators the same, and allowing us to perform the addition:

$$\begin{aligned}\frac{4a}{a-2}+\frac{7a}{-1(a-2)} &= \frac{4a}{a-2}+\frac{-1(7a)}{a-2}\\ &= \frac{4a+(-7a)}{a-2}\\ &= \frac{-3a}{a-2}\end{aligned}$$

Everything is fully factored, so we are done.

ANSWER: $\dfrac{-3a}{a-2}$

Homework 7.4.

Perform the summation, simplify, and state the answer as a quotient of polynomial factorizations.

1. $\dfrac{2}{a+3}+\dfrac{4}{a+3}$

2. $\dfrac{x^2}{x-2}-\dfrac{6x-8}{x-2}$

3. $\dfrac{t^2+4t}{t-1}+\dfrac{2t-7}{t-1}$

4. $\dfrac{a^2+3a}{a^2+5a-6}-\dfrac{4}{a^2+5a-6}$

5. $\dfrac{2x^2+3}{x^2-6x+5}-\dfrac{x^2-5x+9}{x^2-6x+5}$

6. $\dfrac{3}{x}+\dfrac{4}{x^2}$

7. $\dfrac{5}{6r}-\dfrac{5}{8r}$

8. $\dfrac{6}{xy^2}+\dfrac{3}{x^2y}$

9. $\dfrac{8}{9t^3}+\dfrac{5}{6t^2}$

10. $\dfrac{x+5}{8}+\dfrac{x-3}{12}$

11. $\dfrac{a+2}{2}-\dfrac{a-4}{4}$

12. $\dfrac{2a-1}{3a^2}+\dfrac{5a+1}{9a}$

13. $\dfrac{x-1}{4x}-\dfrac{2x+3}{x}$

14. $\dfrac{a+2}{2}-\dfrac{a-4}{4}$

15. $\dfrac{2}{x-1}+\dfrac{2}{x+1}$

16. $\dfrac{2z}{z-1}-\dfrac{3z}{z+1}$

17. $\dfrac{2}{x-5}+\dfrac{3}{4x}$

18. $\dfrac{8x}{x^2-4}-\dfrac{3}{x+2}$

19. $\dfrac{4x}{x^2-25}+\dfrac{x}{x+5}$

20. $\dfrac{t}{t-3}-\dfrac{5}{4t-12}$

21. $\dfrac{2}{x+3}+\dfrac{4}{(x+3)^2}$

22. $\dfrac{2}{5x^2+5x}-\dfrac{4}{3x+3}$

23. $\dfrac{3a}{4a-20}+\dfrac{9a}{6a-30}$

24. $\dfrac{x}{x-5}+\dfrac{x}{5-x}$

25. $\dfrac{t}{y-t}-\dfrac{y}{y+t}$

26. $\dfrac{x}{x-5}+\dfrac{x-5}{x}$

27. $\dfrac{2x}{x^2-1}-\dfrac{4}{x^2+2x-3}$

28. $\dfrac{x-1}{x^2+3x+2}+\dfrac{x+5}{x^2+4x+3}$

29. $\dfrac{x+1}{x^2-2x-35}+\dfrac{x+6}{x^2+7x+10}$

30. $\dfrac{3x+2}{3x+6}+\dfrac{x}{4-x^2}$

31. $\dfrac{4-a^2}{a^2-9}-\dfrac{a-2}{3-a}$

32. $\dfrac{4y}{y^2-1}-\dfrac{2}{y}-\dfrac{2}{y+1}$

33. $\dfrac{x}{x^2+5x+6}-\dfrac{2}{x^2+3x+2}$

34. $\dfrac{2x}{x^2-1}-\dfrac{3}{x^2+5x+4}$

35. $\dfrac{x}{x^2+15x+56}-\dfrac{7}{x^2+13x+42}$

36. $\dfrac{2x}{x^2-9}+\dfrac{5}{x^2+x-6}$

Homework 7.4 Answers.

1. $\dfrac{6}{a+3}$

3. $t+7$

5. $\dfrac{x+6}{x-5}$

7. $\dfrac{5}{24r}$

9. $\dfrac{15t+16}{18t^3}$

11. $\dfrac{a+8}{4}$

13. $\dfrac{-7x-13}{4x}$

15. $\dfrac{4x}{x^2-1}$

17. $\dfrac{11x-15}{4x(x-5)}$

19. $\dfrac{x(x-1)}{(x+5)(x-5)}$

21. $\dfrac{2(x+5)}{(x+3)^2}$

23. $\dfrac{9a}{4(a-5)}$

25. $\dfrac{t^2+2ty-y^2}{(y+t)(y-t)}$

27. $\dfrac{2(x+2)}{(x+1)(x+3)}$

29. $\dfrac{2(x-4)}{(x-7)(x+2)}$

31. $\dfrac{a-2}{(a+3)(a-3)}$

33. $\dfrac{x-3}{(x+3)(x+1)}$

35. $\dfrac{x-8}{(x+8)(x+6)}$

5. Complex Fractions

5.1. Definition.

DEFINITION 5.1.1. Within this text, *complex fractions* are all the expressions which can be built up by combining polynomials with arithmetic operations $+$, $-$, $\cdot$, and $\div$

Many other sources use a narrow definition which only admits fractions of fractions and/or mixed numbers.

The reader should also be careful not to confuse *complex fractions* with fractions which involve complex numbers, which we briefly mention later in the text.

BASIC EXAMPLE 5.1.1. Here are some traditional complex fractions:

$$\frac{1}{2} \div \frac{x}{5}, \quad \frac{1\frac{3}{4}}{6\frac{7}{3}}$$

And here are even more complex fractions which are built up as quotients of sums of rational expressions:

$$\frac{x}{\frac{1}{x}-1}, \quad \frac{\frac{3}{y}-\frac{1}{2}}{ab}, \quad \frac{\frac{x}{y}+\frac{y}{z}}{\frac{1}{a}-\frac{b}{z}}, \quad \frac{1}{1+\frac{1}{1+\frac{1}{2}}}$$

5.2. Simplifying Complex Fractions.

THEOREM 5.2.1. For every complex fraction we can find a rational expression which is *almost* equivalent. That is, the value of the complex fraction will be equal to the value of a corresponding rational expression whenever both values are defined.

PROOF. By induction of arithmetic expressions. The most basic type of complex fraction is a polynomial, which is a rational expression. A sum, a product, or a quotient of rational expressions can be rewritten as a rational expression in simplified form, and they will have the same values, except at finitely many points where one of them is undefined. □

We can simplify every complex fraction by rewriting every sum as a rational expression, and then simplifying products and quotients of rational expressions by canceling common polynomial factors. These steps may have to be repeated several times. For answers, we will factor the rational expressions and state them as quotients of polynomial factorizations.

EXAMPLE 5.2.1. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{x}{1+\dfrac{1}{x}}$$

SOLUTION: We start by rewriting the sum in the denominator as a rational expression. In order to do that, we have to find the LCD for 1 and $1/x$ and then add fraction as usual.

$$\begin{aligned}\frac{x}{1+\dfrac{1}{x}} &= \frac{x}{\dfrac{x}{x}+\dfrac{1}{x}} && \text{the LCD for 1 and } \frac{1}{x} \text{ is } x\\ &= \frac{x}{\left(\dfrac{x+1}{x}\right)}\end{aligned}$$

Now we can divide x by $\dfrac{x+1}{x}$ as usual:

$$\begin{aligned}\frac{x}{\left(\dfrac{x+1}{x}\right)} &= x \div \frac{x+1}{x} && \text{division is multiplication by the reciprocal}\\ &= x \cdot \frac{x}{x+1}\\ &= \frac{x^2}{x+1}\end{aligned}$$

ANSWER: $\dfrac{x^2}{x+1}$

EXAMPLE 5.2.2. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{\dfrac{1}{2}+\dfrac{x}{4}}{\dfrac{1}{4}-\dfrac{x}{2}}$$

SOLUTION: To avoid rewriting a giant fraction, we will simplify this expression in three stages. First we will simplify the sums, and then divide the resulting rational expressions.

Simplify the numerator:

$$\begin{aligned} \frac{1}{2}+\frac{x}{4} &= \frac{2}{4}+\frac{x}{4} && \text{the LCD is 4} \\ &= \frac{2+x}{4} \end{aligned}$$

Simplify the denominator:

$$\begin{aligned} \frac{1}{4}-\frac{x}{2} &= \frac{1}{4}-\frac{2x}{4} && \text{the LCD is 4} \\ &= \frac{1-2x}{4} \end{aligned}$$

Divide numerator by denominator:

$$\begin{aligned} \frac{2+x}{4} \div \frac{1-2x}{4} &= \frac{2+x}{4}\cdot\frac{4}{1-2x} && \text{multiply by reciprocal of divisor} \\ &= \frac{(2+x)4}{4(1-2x)} && \text{show fraction parentheses} \\ &= \frac{2+x}{1-2x} && \text{canceled common factor 4} \end{aligned}$$

ANSWER: $\dfrac{2+x}{1-2x}$

EXAMPLE 5.2.3. Simplify and state the answer as a quotient of polynomial factorizations:

$$1+\frac{2}{3+\frac{4}{x}}$$

SOLUTION: We start by simplifying the sum of rational expressions $3+\dfrac{4}{x}$

$$\begin{aligned} 3+\frac{4}{x} &= \frac{3x}{x}+\frac{4}{x} && \text{the LCD is } x \\ &= \frac{3x+4}{x} \end{aligned}$$

Now we can compute the quotient of rational expressions 2 and $\frac{3x+4}{x}$ in the usual way:

$$2 \div \frac{3x+4}{x} = 2 \cdot \frac{x}{3x+4} = \frac{2x}{3x+4}$$

Finally, we simplify the sum of rational expressions $1 + \frac{2x}{3x+4}$ in the usual way:

$$\begin{aligned} 1+\frac{2x}{3x+4} &= \frac{3x+4}{3x+4}+\frac{2x}{3x+4} && \text{the LCD is } 3x+4 \\ &= \frac{3x+4+(2x)}{3x+4} \\ &= \frac{5x+4}{3x+4} \end{aligned}$$

ANSWER: $\frac{5x+4}{3x+4}$

EXAMPLE 5.2.4. Simplify and state the answer as a quotient of polynomial factorizations:

$$\frac{x^{-1}+x^{-2}}{x^{-3}-x^{-2}}$$

SOLUTION: We can convert negative exponents into rational expressions and then simplify the resulting complex fraction just as before:

$$\frac{x^{-1}+x^{-2}}{x^{-3}-x^{-2}} = \frac{\frac{1}{x}+\frac{1}{x^2}}{\frac{1}{x^3}-\frac{1}{x^2}}$$

Simplify the numerator:

$$\begin{aligned} \frac{1}{x}+\frac{1}{x^2} &= \frac{x}{x^2}+\frac{1}{x^2} && \text{the LCD is } x^2 \\ &= \frac{x+1}{x^2} \end{aligned}$$

Simplify the denominator:

$$\begin{aligned} \frac{1}{x^3}-\frac{1}{x^2} &= \frac{1}{x^3}-\frac{x}{x^3} && \text{the LCD is } x^3 \\ &= \frac{1-x}{x^3} \end{aligned}$$

Divide numerator by denominator:

$$\begin{aligned} \frac{x+1}{x^2} \div \frac{1-x}{x^3} &= \frac{x+1}{x^2} \cdot \frac{x^3}{1-x} \\ &= \frac{(x+1)x^3}{x^2(1-x)} && \frac{x^3}{x^2} = x \\ &= \frac{(x+1)x}{1-x} \end{aligned}$$

ANSWER: $\dfrac{x(x+1)}{1-x}$

Homework 7.5.

Simplify and state the answer as a quotient of polynomial factorizations:

1. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x^2}}$

2. $\dfrac{\dfrac{1}{y^2}-1}{1+\dfrac{1}{y}}$

3. $\dfrac{a-2}{\dfrac{4}{a}-a}$

4. $\dfrac{\dfrac{25}{a}-a}{5+a}$

5. $\dfrac{\dfrac{1}{a^2}-\dfrac{1}{a}}{\dfrac{1}{a^2}+\dfrac{1}{a}}$

6. $\dfrac{\dfrac{1}{b}+\dfrac{1}{2}}{\left(\dfrac{4}{b^2-1}\right)}$

7. $\dfrac{2-\dfrac{4}{x+2}}{5-\dfrac{10}{x+2}}$

8. $\dfrac{4+\dfrac{12}{2x-3}}{5+\dfrac{15}{2x-3}}$

9. $\dfrac{\dfrac{3}{2a-3}+2}{\dfrac{-6}{2a-3}-4}$

10. $\dfrac{\dfrac{-5}{b-5}-3}{\dfrac{10}{b-5}+6}$

11. $\dfrac{\dfrac{x}{x+1}-\dfrac{1}{x}}{\dfrac{x}{x+1}+\dfrac{1}{x}}$

12. $\dfrac{\dfrac{2a}{a-1}-\dfrac{3}{a}}{\dfrac{-6}{a-1}-4}$

13. $\dfrac{\left(\dfrac{3}{x}\right)}{\left(\dfrac{9}{x^2}\right)}$

14. $\dfrac{\left(\dfrac{x}{3x-2}\right)}{\left(\dfrac{x}{9x^2-4}\right)}$

15. $\dfrac{\dfrac{1}{a-h}-\dfrac{1}{a}}{h}$

16. $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

17. $\dfrac{x^{-1}+y^{-1}}{\left(\dfrac{x^2-y^2}{xy}\right)}$

18. $\dfrac{x-x^{-1}}{x+x^{-1}}$

19. $\dfrac{x-1+\dfrac{2}{x-4}}{x+3+\dfrac{6}{x-4}}$

20. $\dfrac{x-5-\dfrac{18}{x+2}}{x+7+\dfrac{6}{x+2}}$

21. $\dfrac{\dfrac{2}{b}-\dfrac{5}{b+3}}{\dfrac{3}{b}+\dfrac{3}{b+3}}$

22. $\dfrac{\dfrac{x-1}{x+1}-\dfrac{x+1}{x-1}}{\dfrac{x-1}{x+1}+\dfrac{x+1}{x-1}}$

Homework 7.5 Answers.

1. $\dfrac{x}{x-1}$

3. $\dfrac{-a}{a+2}$

5. $\dfrac{-a+1}{a+1}$

7. $\dfrac{2}{5}$

9. $-\dfrac{1}{2}$

11. $\dfrac{x^2-x-1}{x^2+x+1}$

13. $\dfrac{x}{3}$

15. $\dfrac{1}{a(a-h)}$

17. $\dfrac{1}{x-y}$

19. $\dfrac{x-2}{x+2}$

21. $\dfrac{-b+2}{2b+3}$

6. Equations with Rational Expressions

6.1. Extraneous Solutions.

DEFINITION 6.1.1 (Extraneous Solution). Some techniques for solving equations will yield numbers which are not solutions. This is a reasonable compromise as long as these techniques never miss a legitimate solution. In this case, we can solve an equation by making a list of all possible solutions, and then checking each one. The ones that make the equation true are the proper solutions, while the rest are called *extraneous solutions*. It is important to understand that strictly speaking they are not solutions at all.

When solving equations involving rational expressions, we will often simplify them by canceling polynomial factors with their reciprocals, for example

$$\frac{(x+1)(x-2)}{x(x-2)} = \frac{x+1}{x}$$

Recall that these expressions are not exactly equivalent, since the one on the left is not defined for $x = 2$. In fact, if $x = 2$, then the reciprocal of $x - 2$ is not defined, and nothing can be canceled. So when we do this within an equation, we make a tacit assumption that the polynomial factor $x - 2 \neq 0$, which is equivalent to $x \neq 2$. In other words, 2 can not be a solution.

THEOREM 6.1.1. An equation involving rational expressions can be solved by multiplying both sides by the polynomial LCD, canceling common polynomial factors, and then solving the resulting polynomial equation. Some or all solutions of the polynomial equation may be extraneous. To find out which ones, it is sufficient to determine which values of the variable make the expressions within the original equation undefined: if they come up as solutions, they must be extraneous, while everything else will work.

A more generic (and somewhat more tedious) way to detect an extraneous solution is to use it in the original equation and show that the equation fails to be satisfied.

EXAMPLE 6.1.1. Solve the equation

$$x - \frac{x}{x-1} = \frac{x-2}{x-1}$$

SOLUTION: Looking at denominators, if $x = 1$ then $x - 1 = 0$, and expressions involving denominator $x - 1$ will have undefined values, so 1 can not be a solution.

Our technique is to multiply both sides of the equation by the LCD, which is $x - 1$, and then simplify fractions:

$$x - \frac{x}{x-1} = \frac{x-2}{x-1} \qquad \text{original equation in factored form}$$

$$\left(x - \frac{x}{x-1}\right)(x-1) = \left(\frac{x-2}{x-1}\right)(x-1) \qquad \text{both sides multiplied by LCD}$$

As we distribute, we end up multiplying each term of the equation by LCD, which amounts to multiplying the numerator of each term by LCD:

$$x(x-1) - \frac{x(x-1)}{(x-1)} = \frac{(x-2)(x-1)}{(x-1)} \qquad \text{distributivity}$$

$$x(x-1) - x = x - 2 \qquad \text{common polynomial factors canceled}$$

This looks like a quadratic equation, so we will solve it by factoring:

$$x^2 - x - x = x - 2 \qquad \text{combine like terms}$$

$$x^2 - 2x = x - 2 \qquad \text{subtract } x \text{ and add 2 on both sides}$$

$$x^2 - 3x + 2 = 0 \qquad \text{factor}$$

$$(x-1)(x-2) = 0$$

By the zero product property, solutions are 1 and 2. Solutions of this quadratic equation are the possible solutions of the original equation involving rational expressions, but some of them may be extraneous. We figured out early on that 1 makes expressions undefined, so it is extraneous, and the only solution is 2.

Alternatively, we can check the answers in the usual way and discard the ones that do not work. When we substitute 1 for x in the original equation, we end up dividing by zero, which is not allowed:

$$(1) - \frac{(1)}{(1)-1} = \frac{(1)-2}{(1)-1}$$

ANSWER: $\{2\}$

EXAMPLE 6.1.2. Solve the equation

$$\frac{1}{2x^2+3x+1} = \frac{1}{x^2-1}$$

SOLUTION: Factor denominators and find the LCD:

$$\begin{aligned} 2x^2+3x+1 &= (2x+1)(x+1) \\ x^2-1 &= (x+1)(x-1) \end{aligned}$$

Recall that LCD is the product of prime polynomial factors, each with the highest exponent we found, and each of these factors has the highest degree 1, so

$$\text{LCD} = (x+1)(x-1)(2x+1)$$

If either of these factors becomes zero, some expressions within the original equation become undefined. We can make a list of forbidden solutions by solving the equation

$$(x+1)(x-1)(2x+1) = 0$$

By the zero product property,

- If $x+1=0$ then $x=-1$
- If $x-1=0$ then $x=1$
- If $2x+1=0$ then $x=-1/2$

So if any of these numbers come up as solutions, we will know they are extraneous. Now we can multiply both sides of the equation by the LCD and get rid of fractions by canceling common polynomial factors:

$$\frac{1}{(2x+1)(x+1)} = \frac{1}{(x+1)(x-1)} \qquad \text{original equation in factored form}$$

Multiply both sides by LCD to get rid of fractions:

$$\left(\frac{1}{(2x+1)(x+1)}\right)(x+1)(x-1)(2x+1) = \left(\frac{1}{(x+1)(x-1)}\right)(x+1)(x-1)(2x+1)$$

Note that denominators do not change, and what we do amounts to multiplying the numerator of each term in the equation by LCD:

$$\begin{aligned} \frac{(x+1)(x-1)(2x+1)}{(2x+1)(x+1)} &= \frac{(x+1)(x-1)(2x+1)}{(x+1)(x-1)} && \text{cancel common factors} \\ x-1 &= 2x+1 && \text{this is a linear equation} \\ -2 &= x && x \text{ is isolated on the right} \end{aligned}$$

The solution −2 of the linear equation is not among the forbidden solutions, so it should work. Alternatively, it can be checked in the original equation involving rational expressions:

$$\begin{aligned}
\frac{1}{2(-2)^2+3(-2)+1} &= \frac{1}{(-2)^2-1}\\
\frac{1}{2\cdot 4+(-6)+1} &= \frac{1}{4-1}\\
\frac{1}{8-6+1} &= \frac{1}{3}\\
\frac{1}{3} &= \frac{1}{3}
\end{aligned}$$

The equation holds true, so −2 is the only solution.

ANSWER: $\{-2\}$

EXAMPLE 6.1.3. Solve the equation

$$x+\frac{24}{x-2}=-9$$

SOLUTION: The LCD is $x-2$, so the only forbidden solution is $x=2$. We multiply both sides by the LCD, distribute as needed, and then simplify resulting rational expressions:

$$\begin{aligned}
\left(x+\frac{24}{x-2}\right)(x-2) &= (-9)(x-2) && \text{both sides multiplied by LCD}\\
x(x-2)+\frac{24(x-2)}{x-2} &= -9x+18 && \text{distributivity}\\
x^2-2x+24 &= -9x+18 && \text{factor } (x-2) \text{ canceled}
\end{aligned}$$

This is a quadratic equation, which we can solve by moving all the non-zero terms to one side, factoring, and applying the zero product property:

$$\begin{aligned}
x^2-2x+24+9x-18 &= -9x+18+9x-18 && \text{added } 9x-18 \text{ to both sides}\\
x^2+7x+6 &= 0 && \text{combined like terms}\\
(x+1)(x+6) &= 0
\end{aligned}$$

By the zero product property, solutions of this quadratic equation are −1 and −6. The only forbidden solution is 2, so both −1 and −6 should work. Alternatively, we can check whether they are solutions for the original equation.

If $x = -1$ then the left side of the original equation evaluates to

$$\begin{aligned} x + \frac{24}{x-2} &= (-1) + \frac{24}{(-1)-2} \\ &= -1 + \frac{24}{-3} \\ &= -1 + (-8) \\ &= -9 \qquad \text{this is the right side, so the equation holds true} \end{aligned}$$

If $x = -6$ then the left side of the original equation evaluates to

$$\begin{aligned} x + \frac{24}{x-2} &= (-6) + \frac{24}{(-6)-2} \\ &= -6 + \frac{24}{-8} \\ &= -6 + (-3) \\ &= -9 \qquad \text{this is the right side, so the equation holds true} \end{aligned}$$

ANSWER: $\{-6, -1\}$

Homework 7.6.

Solve the equation.

1. $\frac{x}{6} - \frac{6}{x} = 0$

2. $\frac{3}{x} = \frac{4}{x} - \frac{1}{5}$

3. $a + \frac{4}{a} = -5$

4. $y + \frac{3}{y} = -4$

5. $\frac{y+3}{y-3} = \frac{6}{y-3}$

6. $\frac{x}{2} = \frac{18}{x}$

7. $3x - \frac{1}{2} - \frac{1}{x} = 0$

8. $x + 1 = \frac{4}{x+1}$

9. $x + \frac{20}{x-4} = \frac{5x}{x-4} - 2$

10. $\frac{x^2+6}{x-1} + \frac{x-2}{x-1} = 2x$

11. $x + \frac{6}{x-3} = \frac{2x}{x-3}$

12. $\frac{x-4}{x-1} = \frac{12}{3-x} + 1$

13. $\frac{2x}{3x-4} = \frac{4x+5}{6x-1} - \frac{3}{3x-4}$

14. $\frac{6x+5}{2x^2-2x} - \frac{2}{1-x^2} = \frac{3x}{x^2-1}$

15. $\frac{3m}{2m-5} - \frac{7}{3m+1} = \frac{3}{2}$

16. $\frac{4x}{2x-6} - \frac{4}{5x-15} = \frac{1}{2}$

17. $\frac{-3}{x+3} = \frac{x}{x+3}$

18. $\frac{3}{2x-6} = \frac{x}{2x-6}$

19. $\frac{4-x}{1-x} = \frac{12}{3-x}$

20. $\frac{7}{3-x} + \frac{1}{2} = \frac{3}{4-x}$

21. $\frac{7}{y-3} - \frac{1}{2} = \frac{y-2}{y-4}$

22. $\frac{2}{3-x} - \frac{6}{8-x} = 1$

23. $\frac{1}{x+2} - \frac{1}{2-x} = \frac{3x+8}{x^2-4}$

24. $\frac{x+2}{3x-1} - \frac{1}{x} = \frac{3x-3}{3x^2-x}$

25. $\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{5}{6}$

26. $\frac{x-1}{x-3} + \frac{x+2}{x+3} = \frac{3}{4}$

27. $\frac{3}{2x+1} + \frac{2x+1}{2x-1} = 1$

28. $\frac{3x-5}{5x-5} + \frac{5x-1}{7x-7} - \frac{x-4}{1-x} = 2$

29. $\frac{3}{x-3} + \frac{5}{x+2} = \frac{5x}{x^2-x-6}$

30. $\frac{2}{x-2} + \frac{1}{x+4} = \frac{x}{x^2+2x-8}$

31. $\frac{5}{y-3} - \frac{30}{y^2-9} = 1$

32. $\frac{1}{x+3}+\frac{1}{x-3}=\frac{1}{x^2-9}$

33. $\frac{5}{x-2}+\frac{3x}{x-2}=\frac{4}{x^2-4x+4}$

34. $\frac{4}{t-3}+\frac{2t}{t-3}=\frac{12}{t^2-6t+9}$

Homework 7.6 Answers.

1. $\{-6, 6\}$

3. $\{-4, -1\}$

5. $\varnothing$

7. $\left\{-\frac{1}{2}, \frac{2}{3}\right\}$

9. $\{3\}$

11. $\{2\}$

13. $\{-1\}$

15. $\{-5\}$

17. $\varnothing$

19. $\{-5, 0\}$

21. $\left\{\frac{16}{3}, 5\right\}$

23. $\{-8\}$

25. $\left\{-\frac{1}{5}, 5\right\}$

27. $\left\{\frac{1}{10}\right\}$

29. $\varnothing$

31. $\{2\}$

33. $\left\{-2, \frac{7}{3}\right\}$

Practice Test 7

Find all variable values which make the given expression undefined:

1. $\dfrac{x+3}{x^2+5x+6}$

2. $\dfrac{y^5}{y^3-10y^2+25y}$

Find the lowest common denominator for the given expressions and state it in a fully factored form:

3. $\dfrac{1}{6}$ $\quad \dfrac{2x}{x+1}$ $\quad \dfrac{4}{2x+2}$

4. $\dfrac{1}{x^2}$ $\quad \dfrac{2}{x^2-1}$ $\quad \dfrac{3}{x^2+2x+1}$

5. $\dfrac{1}{6x^5}$ $\quad \dfrac{1}{4x^7}$ $\quad \dfrac{1}{8x^6}$

6. $\dfrac{7}{10x^2+5x}$ $\quad \dfrac{y}{50x^2}$ $\quad \dfrac{1}{4x^2-1}$

Simplify the expression and state the answer in a fully factored form:

7. $\dfrac{x^2+x-6}{x^2-x-2} \div \dfrac{x^2+10x+21}{x+1}$

8. $\dfrac{x+3}{x-4} - \dfrac{1-2x}{8-2x}$

9. $\dfrac{1}{4x^2-9} + \dfrac{1}{4x^2+12x+9}$

10. $\dfrac{\dfrac{2}{x}-\dfrac{2}{3x}}{\dfrac{1}{x}-\dfrac{5}{6x}}$

11. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x^2}}$

12. $\dfrac{1}{1+\dfrac{1}{1+\frac{1}{x}}}$

Solve the equation:

13. $\dfrac{5}{x} - \dfrac{1}{3} = \dfrac{1}{x}$

14. $2 - \dfrac{1}{x^2+x} = \dfrac{3}{x+1}$

15. $\dfrac{x}{x+2} + \dfrac{2}{x^2+5x+6} = \dfrac{5}{x+3}$

16. $\dfrac{x}{x-5} + \dfrac{3}{x+2} = \dfrac{7x}{x^2-3x-10}$

Practice Test 7 Answers.

1. $\{-2, -3\}$

2. $\{0, 5\}$

3. $6(x+1)$

4. $x^2(x+1)^2(x-1)$

5. $24x^7$

6. $50x^2(2x+1)(2x-1)$

7. $\dfrac{1}{x+7}$

8. $\dfrac{7}{2(x-4)}$

9. $\dfrac{4x}{(2x-3)(2x+3)^2}$

10. 8

11. $\dfrac{x}{x-1}$

12. $\dfrac{x+1}{2x+1}$

13. $\{12\}$

14. $\{-1/2, 1\}$

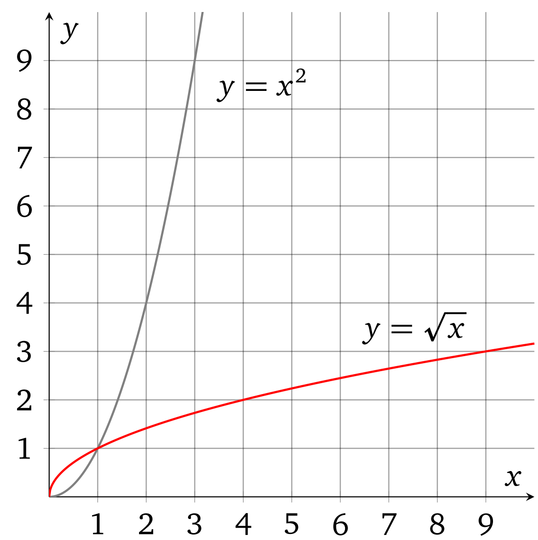
15. $\{4\}$

16. $\{-3\}$

CHAPTER 8

Radicals

1. Introduction to Radicals



1.1. Square Root and Radicals.

DEFINITION 1.1.1 (Principal Square Root). The *principal square root* of a non-negative real number b is the non-negative number y such that $y^2 = b$. It is written like so:

$$\sqrt{b} = y$$

The fancy sign is known as *radical* or *radix* and the expression b is called the *radicand* of the root.

The radical is actually just a traditional notation for exponent:

$$\sqrt{b} = (b)^{\frac{1}{2}}$$

This is perhaps the best way to understand the principal square root, and the reason why it obeys the properties of exponential expressions.

The principal square root of a negative number is not real, but complex.

BASIC EXAMPLE 1.1.1. Here are some easy-to-find square roots.

The principal square root of 16 is 4 because $4^2 = 16$, so we can write $\sqrt{16} = 4$.

$\sqrt{100} = 10$ because $10^2 = 100$.

$\sqrt{0} = 0$ because $0^2 = 0$.

$\sqrt{0.25} = 0.5$ because $0.5^2 = 0.25$.

$\sqrt{\frac{4}{9}} = \frac{2}{3}$ because $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$

$\sqrt{2} = 1.41421\ldots$ is irrational, so it cannot be written as a fraction of integers, and we can only list finitely many digits of its decimal expansion here.

THEOREM 1.1.1. Applying the principal square root to an exponential expression with base $b \geq 0$ amounts to dividing the exponent by 2. For all non-negative real numbers a and all positive even integers k,

$$\sqrt{a^k} = (a)^{\frac{k}{2}}$$

The same is true for exponentiating the principal square root:

$$\left(\sqrt{a}\right)^k = (a)^{\frac{k}{2}}$$

This is true for odd integers as well, as will be shown in the rational exponent section.

BASIC EXAMPLE 1.1.2. When taking roots of exponential expressions, we can check the results by definition.

$\sqrt{x^{14}} = x^{14/2} = x^7$, which can be checked by definition:

$$(x^7)^2 = x^{7\cdot 2} = x^{14}$$

$\sqrt{y^{-6}} = y^{-6/2} = y^{-3} = 1/y^3$, which can be checked by definition:

$$\left(\frac{1}{y^3}\right)^2 = \frac{1}{(y^3)^2} = \frac{1}{y^6} = y^{-6}$$

Exponentiating roots works similarly:

$\left(\sqrt{5}\right)^4 = 5^{4/2} = 5^2 = 25$

$\left(\sqrt{z}\right)^{-20} = z^{-20/2} = z^{-10} = \frac{1}{z^{10}}$

1.2. Principal *n*th Root. The *n*th root is a generalization of the square root. Just like the square root cancels the square for $x \geq 0$

$$\sqrt{x^2} = x$$

the *n*th root will cancel the *n*th power

$$\sqrt[n]{x^n} = x$$

DEFINITION 1.2.1 (Principal *n*th Root). Given a positive *even* integer n, the *principal nth root* of a non-negative real number b is the non-negative number y such that $y^n = b$.

Given a positive *odd* integer n, the *principal nth root* of a real number b is the number y such that $y^n = b$.

The principal *n*th root of b is written like so:

$$\sqrt[n]{b} = y$$

The fancy sign is known as *radical* or *radix*, n is called the *index* and b is called the *radicand* of the root.

Just as was the case with the square root, the radical is actually just a traditional notation for exponent:

$$\sqrt[n]{b} = (b)^{\frac{1}{n}}$$

Note that the principal square root $\sqrt{b}$ is the principal *n*th root for $n = 2$, and could be written as $\sqrt[2]{b}$, but 2 is traditionally not shown.

BASIC EXAMPLE 1.2.1. Here are some easy-to-find *n*th roots.

The principal 3rd root of 125 is 5 because $5^3 = 125$, so we can write $\sqrt[3]{125} = 5$

$\sqrt[5]{0} = 0$ because $0^5 = 0$

$\sqrt[3]{8} = 2$ because $2^3 = 8$

$\sqrt[4]{0.0081} = 0.3$ because $0.3^4 = 0.0081$

$\sqrt[17]{1} = 1$ because $1^{17} = 1$

$\sqrt[3]{\frac{27}{125}} = \frac{3}{5}$ because $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$

BASIC EXAMPLE 1.2.2. Roots with even indices, such as the square root $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, and so on, were only defined for non-negative radicands, but roots with odd index can be applied to negative radicands as well.

$\sqrt[3]{-1000} = -10$ because $(-10)^3 = -1000$.

$\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$

$\sqrt[7]{-1} = -1$ because $(-1)^7 = -1$.

THEOREM 1.2.1. Applying the principal nth root to an exponential expression with base $b \geq 0$ amounts to dividing the exponent by n.

$$\sqrt[n]{b^k} = (b)^{\frac{k}{n}}$$

Similarly for exponentiating the principal nth root:

$$\left(\sqrt[n]{b}\right)^k = (b)^{\frac{k}{n}}$$

BASIC EXAMPLE 1.2.3. When taking roots of exponential expressions, we can check the results by definition.

$\sqrt[6]{x^{12}} = x^{12/6} = x^2$, which can be checked by definition:

$$(x^2)^6 = x^{2\cdot 6} = x^{12}$$

$\sqrt[3]{y^{-15}} = y^{-15/3} = y^{-5} = \frac{1}{y^5}$, which can be checked by definition:

$$\begin{aligned} \left(\frac{1}{y^5}\right)^3 &= \frac{1}{(y^5)^3} \\ &= \frac{1}{y^{15}} \\ &= y^{-15} \end{aligned}$$

Exponentiating roots works similarly:

$\left(\sqrt[4]{3}\right)^{20} = 3^{20/4} = 3^5 = 243$

$\left(\sqrt[5]{z}\right)^{-35} = z^{-35/5} = z^{-7} = \frac{1}{z^7}$

EXAMPLE 1.2.1. Simplify the radical expression, assuming that all variable radicands and bases are non-negative:

$$\sqrt[3]{(a-5)^{18}}$$

SOLUTION:

$$\sqrt[3]{(a-5)^{18}} = (a-5)^{18/3} = (a-5)^6$$

ANSWER: $(a-5)^6$

EXAMPLE 1.2.2. Simplify the radical expression, assuming that all variable radicands and bases are non-negative:

$$\left(\sqrt[4]{x+7}\right)^{-24}$$

SOLUTION: The radicand is a sum $x+7$, so we have to introduce parentheses when we rewrite the radical as an exponent with base $x+7$:

$$\left(\sqrt[4]{x+7}\right)^{-24} = (x+7)^{-24/4} = (x+7)^{-6} = \frac{1}{(x+7)^6}$$

ANSWER: $\dfrac{1}{(x+7)^6}$

Homework 8.1.

Simplify the square root.

1. $\sqrt{81}$

2. $\sqrt{25}$

3. $\sqrt{49}$

4. $\sqrt{16}$

5. $\sqrt{0}$

6. $\sqrt{1}$

Simplify the radical expression.

7. $\sqrt[3]{27}$

8. $\sqrt[3]{64}$

9. $\sqrt[3]{-1000}$

10. $\sqrt[3]{-125}$

11. $\sqrt[3]{-1}$

12. $\sqrt[5]{0}$

13. $\sqrt[5]{32}$

14. $\sqrt[4]{81}$

Simplify the square root, assuming that all variable radicands and bases are non-negative.

15. $\sqrt{x^{12}}$

16. $\sqrt{y^{18}}$

17. $\left(\sqrt{a}\right)^8$

18. $\left(\sqrt{b}\right)^4$

19. $\left(\sqrt{a+6}\right)^6$

20. $\left(\sqrt{b+1}\right)^{10}$

21. $\sqrt{(x-7)^2}$

22. $\sqrt{(y-13)^2}$

23. $\sqrt{(m+k)^{14}}$

24. $\sqrt{(2a-b)^4}$

Simplify the radical expression, assuming that all variable radicands and bases are non-negative.

25. $\sqrt[3]{x^{30}}$

26. $\sqrt[3]{y^{12}}$

27. $\sqrt[5]{a^{20}}$

28. $\sqrt[5]{b^{10}}$

29. $\left(\sqrt[3]{a}\right)^{18}$

30. $\left(\sqrt[5]{u}\right)^{25}$

31. $\sqrt[4]{x^{2000}}$

32. $\sqrt[4]{u^4}$

33. $\sqrt{x^{-12}}$

34. $\sqrt{b^{-16}}$

35. $\left(\sqrt[4]{y}\right)^{-20}$

36. $\left(\sqrt[4]{k}\right)^{-4}$

Homework 8.1 Answers.

1. 9

3. 7

5. 0

7. 3

9. -10

11. -1

13. 2

15. x^6

17. a^4

19. $(a+6)^3$

21. $x-7$

23. $(m+k)^7$

25. x^{10}

27. a^4

29. a^6

31. x^{500}

33. $\frac{1}{x^6}$

35. $\frac{1}{y^5}$

2. Products with Square Roots

2.1. Distributivity Over Multiplication.

THEOREM 2.1.1. Principal square root distributes over multiplication. For all non-negative real numbers x and y

$$\sqrt{xy} \quad = \quad \sqrt{x} \cdot \sqrt{y}$$

This property extends naturally to a product with any number of factors.

PROOF. Using the exponential representation of the root and the properties of the exponent:

$$\sqrt{xy} = (xy)^{\frac{1}{2}} = (x)^{\frac{1}{2}}(y)^{\frac{1}{2}} = \sqrt{x} \cdot \sqrt{y}$$

□

We are assuming all variable bases and radicands to be non-negative throughout this chapter because several very nice properties fail otherwise. In particular, the distributive property $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$ works for positive radicands, but allowing negative radicands leads to nonsense like

$$\begin{aligned} 1 &= \sqrt{1} \\ &= \sqrt{(-1)(-1)} \\ &= \sqrt{-1}\sqrt{-1} && \text{illegitimate distributivity} \\ &= -1 \end{aligned}$$

The mistake above is in the third equals sign: there is nothing wrong with the complex number $\sqrt{-1}$, it is just that the distributive property does not apply to negative radicands.

EXAMPLE 2.1.1. Simplify the expression $\sqrt{8}$

SOLUTION: When simplifying expressions involving square roots, it is customary to simplify the radicand by taking square roots of factors which happen to be perfect squares. So $\sqrt{8}$ will be written in the answer as $2\sqrt{2}$ because

$$\begin{aligned} \sqrt{8} &= \sqrt{4 \cdot 2} \\ &= \sqrt{4} \cdot \sqrt{2} \\ &= 2 \cdot \sqrt{2} \end{aligned}$$

ANSWER: $2\sqrt{2}$

EXAMPLE 2.1.2. Simplify the expression $\sqrt{75}$

SOLUTION: We simplify the radicand by finding the largest integer factor which happens to be a perfect square, and taking its square root separately by distributivity:

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5 \cdot \sqrt{3}$$

ANSWER: $5\sqrt{3}$

EXAMPLE 2.1.3. Simplify the expression, assuming that all variable bases and radicands are non-negative:

$$\sqrt{x^3}$$

SOLUTION: We simplify the radicand by finding the largest even integer exponent of x, and taking its square root separately by distributivity:

$$\begin{aligned} \sqrt{x^3} &= \sqrt{x^2 \cdot x} \\ &= \sqrt{x^2} \cdot \sqrt{x} \\ &= x \cdot \sqrt{x} \end{aligned}$$

ANSWER: $x\sqrt{x}$

EXAMPLE 2.1.4. Simplify the expression, assuming that all variable bases and radicands are non-negative:

$$\sqrt{x^{17}}$$

SOLUTION: The largest even exponent below 17 is 16:

$$\begin{aligned} \sqrt{x^{17}} &= \sqrt{x^{16} \cdot x} \\ &= \sqrt{x^{16}} \cdot \sqrt{x} \\ &= x^{16/2} \cdot \sqrt{x} && \text{simplified square root} \\ &= x^8 \cdot \sqrt{x} \end{aligned}$$

ANSWER: $x^8\sqrt{x}$

EXAMPLE 2.1.5. Simplify the expression, assuming that all variable bases and radicands are non-negative:

$$\sqrt{2x^3}\cdot\sqrt{8x}$$

SOLUTION: Here it helps to simplify the product under the square root before simplifying the radicand:

$$\begin{aligned}
\sqrt{2x^3}\cdot\sqrt{8x} &= \sqrt{2x^3\cdot 8x} && \text{distributivity}\\
&= \sqrt{16x^4} &&\\
&= \sqrt{16}\cdot\sqrt{x^4} && \text{distributivity the other way}\\
&= 4\cdot x^{4/2} && \text{simplify square roots}\\
&= 4\cdot x^2 &&
\end{aligned}$$

ANSWER: $4x^2$

EXAMPLE 2.1.6. Simplify the expression, assuming that all variable bases and radicands are non-negative:

$$2\sqrt{6}\left(\sqrt{3}+5\sqrt{2}\right)$$

SOLUTION: It is important not to confuse the distributivity of multiplication over addition with the distributivity of the principal root over multiplication.

$$\begin{aligned}
2\sqrt{6}\left(\sqrt{3}+5\sqrt{2}\right) &= 2\sqrt{6}\cdot\sqrt{3}+2\sqrt{6}\cdot 5\sqrt{2} && \text{distributivity of } \cdot \text{ over } +\\
&= 2\sqrt{6\cdot 3}+10\sqrt{6\cdot 2} && \text{distributivity of } \sqrt{\ } \text{ over } \cdot\\
&= 2\sqrt{18}+10\sqrt{12} &&
\end{aligned}$$

Now we simplify the radicands:

$$\begin{aligned}
2\sqrt{18}+10\sqrt{12} &= 2\sqrt{9\cdot 2}+10\sqrt{4\cdot 3} && \text{finding perfect square factors}\\
&= 2\sqrt{9}\cdot\sqrt{2}+10\sqrt{4}\cdot\sqrt{3} && \text{distributivity of } \sqrt{\ } \text{ over } \cdot\\
&= 2\cdot 3\cdot\sqrt{2}+10\cdot 2\cdot\sqrt{3} && \text{simplified square roots}\\
&= 6\sqrt{2}+20\sqrt{3} &&
\end{aligned}$$

ANSWER: $6\sqrt{2}+20\sqrt{3}$

EXAMPLE 2.1.7. Simplify the expression, assuming that all variable bases and radicands are non-negative:

$$-5x\sqrt{x^3y^4}\cdot(-2)\sqrt{xy^5}$$

SOLUTION: We will combine the two radicands into one using distributivity, and then simplify the result. First, we change the order of multiplication:

$$\begin{aligned}
-5x\sqrt{x^3y^4}\cdot(-2)\sqrt{xy^5} &= (-5)(-2)\cdot x\cdot\sqrt{x^3y^4}\cdot\sqrt{xy^5} & \\
&= 10x\sqrt{x^3y^4\cdot xy^5} & \text{distributivity of } \sqrt{\ } \text{ over } \cdot \\
&= 10x\sqrt{x^{3+1}y^{4+5}} & \text{properties of exponent} \\
&= 10x\sqrt{x^4y^9} &
\end{aligned}$$

Now we can simplify the radicand:

$$\begin{aligned}
10x\sqrt{x^4y^9} &= 10x\sqrt{x^4\cdot y^8\cdot y} & \text{find largest even power of } y \\
&= 10x\sqrt{x^4}\cdot\sqrt{y^8}\cdot\sqrt{y} & \text{distributivity} \\
&= 10x\cdot x^{4/2}\cdot y^{8/2}\cdot\sqrt{y} & \text{simplify square roots} \\
&= 10x\cdot x^2\cdot y^4\cdot\sqrt{y} & \\
&= 10x^{1+2}\cdot y^4\sqrt{y} & \text{properties of exponent} \\
&= 10x^3\cdot y^4\cdot\sqrt{y} &
\end{aligned}$$

ANSWER: $10x^3y^4\sqrt{y}$

Homework 8.2.

Simplify the expression, assuming that all variable bases and radicands are non-negative.

1. $\sqrt{12}$

2. $\sqrt{18}$

3. $\sqrt{27}$

4. $\sqrt{60}$

5. $\sqrt{72}$

6. $\sqrt{80}$

7. $\sqrt{20x^5}$

8. $\sqrt{32y^7}$

9. $\sqrt{a^{10}b^{13}}$

10. $\sqrt{49x^{49}}$

11. $3\sqrt{5}\cdot(-4)\sqrt{16}$

12. $-5\sqrt{10}\cdot\sqrt{15}$

13. $\sqrt{12m}\cdot\sqrt{15m}$

14. $\sqrt{5y^3}\cdot(-5)\sqrt{10y^2}$

15. $\sqrt{6}(\sqrt{2}+2)$

16. $\sqrt{10}(\sqrt{5}+\sqrt{2})$

17. $\sqrt{10b}\cdot\sqrt{50b}$

18. $\sqrt{6a}\cdot\sqrt{18a^7}$

19. $\sqrt{x^2y^3}\cdot\sqrt{xy^4}$

20. $\sqrt{a^3b^2}\cdot\sqrt{ab}$

21. $5\sqrt{10}(5n+\sqrt{2})$

22. $\sqrt{15}(\sqrt{5}-3\sqrt{3x})$

23. $\sqrt{6xy}\cdot\sqrt{12x^2y^5}$

24. $\sqrt{5ab^5}\cdot\sqrt{20a^2b^3}$

Homework 8.2 Answers.

1. $2\sqrt{3}$

3. $3\sqrt{3}$

5. $6\sqrt{2}$

7. $2x^2\sqrt{5x}$

9. $a^5b^6\sqrt{b}$

11. $-48\sqrt{5}$

13. $6m\sqrt{5}$

15. $2\sqrt{3}+2\sqrt{6}$

17. $10b\sqrt{5}$

19. $xy^3\sqrt{xy}$

21. $25n\sqrt{10}+10\sqrt{5}$

23. $6xy^3\sqrt{2x}$

3. Quotients with Square Roots

3.1. Distributivity Over Division.

THEOREM 3.1.1. Principal square root distributes over division. For all non-negative real numbers a and all positive reals c

$$\sqrt{\frac{a}{c}} = \frac{\sqrt{a}}{\sqrt{c}}$$

This results generalizes naturally to fractions with any number of factors.

PROOF. Using the exponential representation of the root and the properties of the exponent:

$$\sqrt{\frac{x}{y}} = \left(\frac{x}{y}\right)^{\frac{1}{2}} = \frac{(x)^{\frac{1}{2}}}{(y)^{\frac{1}{2}}} = \frac{\sqrt{x}}{\sqrt{y}}$$

□

As we simplify quotients with radicals, we will state all answers in a form where radicals are confined to numerators, and radicands are fully simplified.

DEFINITION 3.1.1. We say that the *denominator is rationalized* in a fraction if radicals appear in the numerator only, and radicands are simplified.

EXAMPLE 3.1.1. Simplify and rewrite with rationalized denominator:

$$\frac{1}{\sqrt{3}}$$

SOLUTION: The radicand is simplified, but it is in the wrong place. To get rid of the radical in the monomial denominator, we can multiply the expression by the radical over itself, in this case by $\sqrt{3}/\sqrt{3}$:

$$\begin{aligned} \frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} && \text{but } \sqrt{3} \cdot \sqrt{3} = \sqrt{3 \cdot 3} = \sqrt{3^2} = 3 \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

ANSWER: $\frac{\sqrt{3}}{3}$

EXAMPLE 3.1.2. Simplify and rewrite with rationalized denominator:

$$\frac{2x}{\sqrt{6x}}$$

SOLUTION: As before, we can rationalize the denominator by multiplying both numerator and denominator by the radical we are trying to eliminate:

$$\begin{aligned}
\frac{2x}{\sqrt{6x}} &= \frac{2x}{\sqrt{6x}} \cdot \frac{\sqrt{6x}}{\sqrt{6x}} \\
&= \frac{2x \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}} \\
&= \frac{2x\sqrt{6x}}{\sqrt{(6x)^2}} && \text{distributivity of } \sqrt{\ } \text{ over } \cdot \\
&= \frac{2x\sqrt{6x}}{6x} && \text{common factors 2 and } x \text{ cancel} \\
&= \frac{\sqrt{6x}}{3}
\end{aligned}$$

ANSWER: $\frac{\sqrt{6x}}{3}$

EXAMPLE 3.1.3. Simplify and rewrite with rationalized denominator:

$$\frac{\sqrt{20x^5}}{\sqrt{5x^3}}$$

SOLUTION: Here it makes more sense to apply distributivity and simplify the fraction under the radical sign before rationalizing anything:

$$\begin{aligned}
\frac{\sqrt{20x^5}}{\sqrt{5x^3}} &= \sqrt{\frac{20x^5}{5x^3}} && \text{distributivity of } \sqrt{\ } \text{ over } \div \\
&= \sqrt{4x^2} \\
&= 2x
\end{aligned}$$

ANSWER: $2x$

Homework 8.3.

Simplify and rewrite with rationalized denominator.

1. $\dfrac{\sqrt{500}}{\sqrt{5}}$

2. $\dfrac{\sqrt{72}}{\sqrt{2}}$

3. $\dfrac{\sqrt{50}}{\sqrt{2}}$

4. $\dfrac{\sqrt{40}}{\sqrt{10}}$

5. $\dfrac{\sqrt{55}}{\sqrt{5}}$

6. $\dfrac{\sqrt{18}}{\sqrt{3}}$

7. $\dfrac{\sqrt{5}}{\sqrt{20}}$

8. $\dfrac{\sqrt{2}}{\sqrt{18}}$

9. $\sqrt{\dfrac{4}{25}}$

10. $\sqrt{\dfrac{9}{49}}$

11. $\sqrt{\dfrac{2a^5}{50a}}$

12. $\sqrt{\dfrac{25}{64}}$

13. $\sqrt{\dfrac{6x^7}{32x}}$

14. $\sqrt{\dfrac{10y^9}{18y^5}}$

15. $\sqrt{\dfrac{4x^3}{50x}}$

16. $\sqrt{\dfrac{100}{49}}$

17. $\dfrac{\sqrt{36}}{\sqrt{6}}$

18. $\dfrac{\sqrt{18}}{\sqrt{32}}$

19. $\dfrac{\sqrt{63y^3}}{\sqrt{7y}}$

20. $\dfrac{\sqrt{8x}}{\sqrt{2x}}$

21. $\dfrac{\sqrt{9b}}{\sqrt{3b}}$

22. $\dfrac{\sqrt{48x^5}}{\sqrt{3x^3}}$

23. $\dfrac{5}{\sqrt{7}}$

24. $\dfrac{2}{\sqrt{3}}$

25. $\dfrac{\sqrt{25}}{\sqrt{8}}$

26. $\dfrac{\sqrt{16}}{\sqrt{27}}$

27. $\dfrac{\sqrt{3}}{\sqrt{50}}$

28. $\dfrac{\sqrt{5}}{\sqrt{7}}$

29. $\dfrac{\sqrt{3a}}{\sqrt{32}}$

30. $\dfrac{\sqrt{2a}}{\sqrt{45}}$

31. $\dfrac{\sqrt{6}}{\sqrt{5}}$

32. $\dfrac{\sqrt{5}}{\sqrt{18}}$

33. $\dfrac{\sqrt{27x^5}}{\sqrt{3x}}$

34. $\dfrac{\sqrt{21a^9}}{\sqrt{7a^3}}$

35. $\dfrac{\sqrt{35x^{13}}}{\sqrt{5x^5}}$

36. $\dfrac{\sqrt{7b^5}}{\sqrt{28b}}$

Homework 8.3 Answers.

1. 10

3. 5

5. $\sqrt{11}$

7. $\frac{1}{2}$

9. $\frac{2}{5}$

11. $\frac{a^2}{5}$

13. $\frac{x^3\sqrt{3}}{4}$

15. $\frac{x\sqrt{2}}{5}$

17. $\sqrt{6}$

19. $3y$

21. $\sqrt{3}$

23. $\frac{5\sqrt{7}}{7}$

25. $\frac{5\sqrt{2}}{4}$

27. $\frac{\sqrt{6}}{10}$

29. $\frac{\sqrt{6a}}{8}$

31. $\frac{\sqrt{30}}{5}$

33. $3x^2$

35. $x^4\sqrt{7}$

4. Sums with Square Roots

4.1. Radical Like Terms. Adding terms with square roots is possible using the distributivity of multiplication over addition.

DEFINITION 4.1.1. Terms involving radicals are *like* (or *similar*) if they have the same variable factors with the same respective exponents, and the same radical factors with the same respective radicands.

BASIC EXAMPLE 4.1.1. Some pairs of like terms:

$x\sqrt{7}, \quad 4x\sqrt{7}$

$\sqrt{x}, \quad 5\sqrt{x}$

$x\sqrt{x}, \quad -6x\sqrt{x}$

$x^2y\sqrt{2ab}, \quad 10x^2y\sqrt{2ab}$

BASIC EXAMPLE 4.1.2. Some pairs of terms which are not like:

$y\sqrt{y}, \quad \sqrt{y}$ are not like because they have different exponents for y variable.

$\sqrt{2x}, \quad \sqrt{5x}$ are not like because they have different radicands.

$\sqrt{8}, \quad 3\sqrt{2}$ are not like because they have different radicands, but they will be like if we simplify the radicands:

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

$2\sqrt{2}$ and $3\sqrt{2}$ are indeed like.

EXAMPLE 4.1.1. Simplify the expression

$$4\sqrt{6} - 5\sqrt{2} - \sqrt{6}$$

SOLUTION: The terms $4\sqrt{6}$ and $-\sqrt{6}$ are like, so we can combine them:

$$\begin{aligned} 4\sqrt{6} - 5\sqrt{2} - \sqrt{6} &= 4\sqrt{6} - \sqrt{6} - 5\sqrt{2} && \text{terms can be added in any order} \\ &= (4-1)\sqrt{6} - 5\sqrt{2} && \text{distributivity} \\ &= 3\sqrt{6} - 5\sqrt{2} \end{aligned}$$

The remaining terms are not like, so we cannot simplify this expression any further.

ANSWER: $3\sqrt{6} - 5\sqrt{2}$

EXAMPLE 4.1.2. Simplify the expression

$$8\sqrt{18}-7\sqrt{2}$$

SOLUTION: These terms are not like, but the radicand 18 can be simplified:

$$\begin{aligned} 8\sqrt{18}-7\sqrt{2} &= 8\sqrt{9\cdot 2}-7\sqrt{2} \\ &= 8\sqrt{9}\sqrt{2}-7\sqrt{2} \\ &= 8\cdot 3\sqrt{2}-7\sqrt{2} \\ &= 24\sqrt{2}-7\sqrt{2} \end{aligned}$$

These terms are like, so we can combine them:

$$\begin{aligned} 24\sqrt{2}-7\sqrt{2} &= (24-7)\sqrt{2} \\ &= 17\sqrt{2} \end{aligned}$$

ANSWER: $17\sqrt{2}$

EXAMPLE 4.1.3. Simplify the expression

$$(3+\sqrt{7})(2-5\sqrt{7})$$

SOLUTION: We begin by multiplying the sums using the standard binomial pattern, where we multiply each term of the first sum by each term of the second sum, and then add all of these products together:

$$\begin{aligned} (3+\sqrt{7})(2-5\sqrt{7}) &= 3\cdot 2-3\cdot 5\sqrt{7}+\sqrt{7}\cdot 2-\sqrt{7}\cdot 5\sqrt{7} \\ &= 6-15\sqrt{7}+2\sqrt{7}-5\sqrt{7}\sqrt{7} && \text{simplified products} \\ &= 6-15\sqrt{7}+2\sqrt{7}-5\cdot 7 && \text{because } \left(\sqrt{7}\right)^2=7 \\ &= 6-15\sqrt{7}+2\sqrt{7}-35 \end{aligned}$$

We have two pairs of like terms here: the two integers as well as $-15\sqrt{7}$ and $2\sqrt{7}$.

$$\begin{aligned} 6-15\sqrt{7}+2\sqrt{7}-35 &= 6-35-15\sqrt{7}+2\sqrt{7} \\ &= -29+(-15+2)\sqrt{7} \\ &= -29-13\sqrt{7} \end{aligned}$$

ANSWER: $-29-13\sqrt{7}$

4.2. Rationalizing Sums. A sum with two terms can be rationalized if it is multiplied by its *conjugate*, which is the sum of the first term and the opposite of the second term.

DEFINITION 4.2.1. For all real numbers A, B, C, D, the sums

$$A\sqrt{B}+C\sqrt{D}$$

and

$$A\sqrt{B}-C\sqrt{D}$$

are the *radical conjugates* (or simply *conjugates*) of each other.

THEOREM 4.2.1 (Rationalizing the Denominator). In a fraction, a sum with two terms in the numerator or the denominator can be rationalized by multiplying both the numerator and the denominator by the conjugate of that sum.

PROOF. We will prove this statement for a sum of square roots in the denominator. If the denominator is $a\sqrt{b}+c\sqrt{d}$, then its conjugate is $a\sqrt{b}-c\sqrt{d}$.

$$\begin{aligned}\frac{1}{a\sqrt{b}+c\sqrt{d}} &= \frac{1}{a\sqrt{b}+c\sqrt{d}}\cdot\frac{a\sqrt{b}-c\sqrt{d}}{a\sqrt{b}-c\sqrt{d}}\\ &= \frac{a\sqrt{b}-c\sqrt{d}}{(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})}\end{aligned}$$

Notice that denominators are sums, and we had to show the invisible fraction parentheses when we multiplied the denominators. Notice also that the denominator now is a difference of squares, and can be simplified:

$$\begin{aligned}\frac{a\sqrt{b}-c\sqrt{d}}{(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})} &= \frac{a\sqrt{b}-c\sqrt{d}}{\left(a\sqrt{b}\right)^2-\left(c\sqrt{d}\right)^2}\\ &= \frac{a\sqrt{b}-c\sqrt{d}}{a^2\left(\sqrt{b}\right)^2-c^2\left(\sqrt{d}\right)^2}\\ &= \frac{a\sqrt{b}-c\sqrt{d}}{a^2b-c^2d}\end{aligned}$$

The new denominator is free of square roots now, even though they moved into the numerator.

The case with the difference of square roots, as well as the cases with rationalizing the numerator have similar proofs. □

EXAMPLE 4.2.1. Rationalize the denominator:

$$\frac{6}{3+\sqrt{5}}$$

SOLUTION: The conjugate of $3+\sqrt{5}$ is $3-\sqrt{5}$:

$$\begin{aligned}
\frac{6}{3+\sqrt{5}} &= \frac{6}{3+\sqrt{5}}\cdot\frac{3-\sqrt{5}}{3-\sqrt{5}} \\
&= \frac{6(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} && \text{difference of squares in the denominator} \\
&= \frac{6(3-\sqrt{5})}{3^2-\left(\sqrt{5}\right)^2} \\
&= \frac{6(3-\sqrt{5})}{9-5} && \text{simplify the denominator} \\
&= \frac{6(3-\sqrt{5})}{4} && \text{cancel common factor 2} \\
&= \frac{3(3-\sqrt{5})}{2}
\end{aligned}$$

ANSWER: $\dfrac{3(3-\sqrt{5})}{2}$

EXAMPLE 4.2.2. Rationalize the denominator:

$$\frac{5x}{\sqrt{x}-\sqrt{y}}$$

SOLUTION: The conjugate of $\sqrt{x}-\sqrt{y}$ is $\sqrt{x}+\sqrt{y}$:

$$\begin{aligned}
\frac{5x}{\sqrt{x}-\sqrt{y}} &= \frac{5x}{\sqrt{x}-\sqrt{y}}\cdot\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} \\
&= \frac{5x(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}
\end{aligned}$$

The denominator is now a difference of squares, so we can simplify it:

$$\frac{5x(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} = \frac{5x(\sqrt{x}+\sqrt{y})}{\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2}$$

$$= \frac{5x(\sqrt{x}+\sqrt{y})}{x-y}$$

ANSWER: $\dfrac{5x(\sqrt{x}+\sqrt{y})}{x-y}$

Homework 8.4.

Simplify the expression.

1. $2\sqrt{5}+2\sqrt{5}+2\sqrt{5}$

2. $-3\sqrt{6}-3\sqrt{3}-2\sqrt{3}$

3. $-3\sqrt{2}+3\sqrt{5}+3\sqrt{5}$

4. $-2\sqrt{6}-\sqrt{3}-3\sqrt{6}$

5. $-2\sqrt{6}-2\sqrt{6}-\sqrt{6}$

6. $-3\sqrt{3}+2\sqrt{3}-2\sqrt{3}$

7. $3\sqrt{5}+3\sqrt{5}+2\sqrt{5}$

8. $-\sqrt{5}+2\sqrt{3}-2\sqrt{3}$

9. $2\sqrt{2}-3\sqrt{18}-\sqrt{2}$

10. $-\sqrt{54}-3\sqrt{6}+3\sqrt{27}$

11. $-3\sqrt{6}-\sqrt{12}+4\sqrt{3}$

12. $-\sqrt{5}-\sqrt{5}-2\sqrt{45}$

13. $3\sqrt{2}+2\sqrt{8}-3\sqrt{18}$

14. $2\sqrt{20}+2\sqrt{20}-\sqrt{3}$

15. $3\sqrt{18}-\sqrt{2}-3\sqrt{2}$

16. $-3\sqrt{27}+2\sqrt{3}-\sqrt{12}$

17. $-3\sqrt{6}-3\sqrt{6}-\sqrt{3}+3\sqrt{6}$

18. $-2\sqrt{2}-\sqrt{2}+3\sqrt{8}+3\sqrt{6}$

19. $-2\sqrt{18}-3\sqrt{8}-\sqrt{20}+2\sqrt{20}$

20. $-3\sqrt{18}-\sqrt{8}+2\sqrt{8}+2\sqrt{8}$

21. $\sqrt{300x}-\sqrt{3x}$

22. $\sqrt{2x}+\sqrt{50x}$

23. $4x\sqrt{x}-\sqrt{x^3}$

24. $2y\sqrt{y}+3\sqrt{y^3}$

25. $(\sqrt{5}-5)(2\sqrt{5}-1)$

26. $(-2+\sqrt{3})(-5+2\sqrt{3})$

27. $(\sqrt{2}+\sqrt{6})(2\sqrt{6}-3\sqrt{2})$

28. $(4\sqrt{5}-\sqrt{20})(\sqrt{20}-\sqrt{5})$

Rationalize the denominator and simplify.

29. $\dfrac{4}{3+\sqrt{5}}$

30. $\dfrac{-4}{4-4\sqrt{2}}$

31. $\dfrac{1}{\sqrt{2}+1}$

32. $\dfrac{3}{\sqrt{6}-2}$

33. $\dfrac{2}{5+\sqrt{2}}$

34. $\dfrac{3}{4-3\sqrt{3}}$

35. $\dfrac{5}{\sqrt{3}+4\sqrt{5}}$

36. $\dfrac{5}{2\sqrt{3}-\sqrt{2}}$

37. $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

38. $\dfrac{2}{2\sqrt{5}+2\sqrt{3}}$

Homework 8.4 Answers.

1. $6\sqrt{5}$

3. $-3\sqrt{2}+6\sqrt{5}$

5. $-5\sqrt{6}$

7. $8\sqrt{5}$

9. $-8\sqrt{2}$

11. $-3\sqrt{6}+2\sqrt{3}$

13. $-2\sqrt{2}$

15. $5\sqrt{2}$

17. $-3\sqrt{6}-\sqrt{3}$

19. $-12\sqrt{2}+2\sqrt{5}$

21. $9\sqrt{3x}$

23. $3x\sqrt{x}$

25. $15-11\sqrt{5}$

27. $6-2\sqrt{3}$

29. $3-\sqrt{5}$

31. $\sqrt{2}-1$

33. $\dfrac{2(5-\sqrt{2})}{23}$

35. $\dfrac{-5(\sqrt{3}-4\sqrt{5})}{77}$

37. $\sqrt{x}+\sqrt{y}$

5. Equations with Square Roots

5.1. Squaring Both Sides. Some equations involving principal square roots can be solved by squaring both sides. This operation does not always produce equivalent equations, and will create extraneous solutions every now and then.

THEOREM 5.1.1. If $A = B$, then the solution set of the equation $A^2 = B^2$ includes all of the solutions of $A = B$, but not necessarily the other way around. This means we can solve the original equation by solving $A^2 = B^2$, and then discarding extraneous solutions, if any.

BASIC EXAMPLE 5.1.1. It is easy to see that squaring both sides may create additional solutions. The solution set of the equation

$$x = 1$$

is $\{1\}$, while the solution set of

$$(x)^2 = (1)^2$$

which is equivalent to

$$x^2 = 1$$

is $\{1, -1\}$.

EXAMPLE 5.1.1. Determine which of the given numbers are solutions for the equation:

$$x - 5 = \sqrt{x + 7}$$

$$2, \quad 9$$

SOLUTION: If $x = 2$ then

$$\begin{aligned} 2 - 5 &= \sqrt{2 + 7} \\ -3 &= 3 \end{aligned}$$

This is false, so 2 is not a solution, but an *extraneous solution*.

If $x = 9$ then

$$\begin{aligned} 9 - 5 &= \sqrt{9 + 7} \\ 4 &= 4 \end{aligned}$$

This is true, so 9 is a solution.

ANSWER: $\{9\}$

EXAMPLE 5.1.2. Solve the equation $\sqrt{x-6}-3=0$

SOLUTION: We want to solve the equation for the term with the radical, or else squaring both sides will be pointless.

$$\begin{aligned} \sqrt{x-6}-3 &= 0 \\ \sqrt{x-6} &= 3 && \text{added 3 to both sides} \\ \left(\sqrt{x-6}\right)^2 &= (3)^2 \\ x-6 &= 9 \end{aligned}$$

This looks like a linear equation, which we solve by isolating the variable:

$$\begin{aligned} x-6 &= 9 \\ x &= 9+6 && \text{added 6 to both sides} \\ x &= 15 \end{aligned}$$

The solution checks out:

$$\sqrt{15-6}-3=\sqrt{9}-3=3-3=0$$

ANSWER: $\{15\}$

EXAMPLE 5.1.3. Solve the equation $6+2\sqrt{x-4}=0$

SOLUTION: We start by solving for the term with the radical:

$$\begin{aligned} 6+2\sqrt{x-4} &= 0 && \text{subtract 6 on both sides} \\ 2\sqrt{x-4} &= -6 && \text{divide both sides by 2 to simplify} \\ \sqrt{x-4} &= -3 && \text{square both sides} \\ \left(\sqrt{x-4}\right)^2 &= (-3)^2 \\ x-4 &= 9 && \text{add 4 to both sides} \\ x &= 13 \end{aligned}$$

Check the solution:

$$\begin{aligned} 6+2\sqrt{13-4} &= 0 \\ 6+2\sqrt{9} &= 0 \\ 6+2\cdot 3 &= 0 \\ 12 &= 0 \end{aligned}$$

This is false, so 13 is an extraneous solution, and since there are no others, the solution set is empty.

ANSWER: $\varnothing$

In the previous example, we could have saved ourselves some time if we used the fact that the principal square root is never negative. We could have concluded there are no solutions from

$$2\sqrt{x-4} = -6$$

which is impossible to satisfy, since the left side is non-negative, while the right side is negative.

THEOREM 5.1.2. The principal square root is never negative. For any real number b,

- if b is positive, then $\sqrt{b} > 0$
- if $b = 0$, then $\sqrt{b} = \sqrt{0} = 0$
- if b is negative, then the principal square root of b is not real, but complex.

EXAMPLE 5.1.4. Solve the equation

$$\sqrt{x^2-4} + 4 = 1$$

SOLUTION: We start by solving for the term with the radical:

$$\begin{aligned} \sqrt{x^2-4}+4 &= 1 \\ \sqrt{x^2-4} &= -3 \end{aligned}$$

We notice that the left side is non-negative, while the right side is negative, so there are no solutions possible.

ANSWER: $\varnothing$

EXAMPLE 5.1.5. Solve the equation

$$\sqrt{3-4x} - \sqrt{-7x-6} = 0$$

SOLUTION: The left side is a sum of two terms, and squaring a sum should be avoided when possible. But if we isolate the radicals, one on each side of the equation, then we can get rid of both of them at the same time:

$$\begin{aligned}
\sqrt{3-4x}-\sqrt{-7x-6} &= 0 \\
\sqrt{3-4x} &= \sqrt{-7x-6} && \text{add } \sqrt{-7x-6} \text{ to both sides} \\
\left(\sqrt{3-4x}\right)^2 &= \left(\sqrt{-7x-6}\right)^2 && \text{square both sides} \\
3-4x &= -7x-6 && \text{squares cancel square roots} \\
3x &= -9 && \text{isolate the term with } x \text{ on the left} \\
x &= -3
\end{aligned}$$

Check whether -3 is a solution:

$$\begin{aligned}
\sqrt{3-4x}-\sqrt{-7x-6} &= \sqrt{3-4(-3)}-\sqrt{-7(-3)-6} \\
&= \sqrt{3+12}-\sqrt{21-6} \\
&= \sqrt{15}-\sqrt{15} \\
&= 0
\end{aligned}$$

The equation holds, so -3 is the only solution.

ANSWER: $\{-3\}$

EXAMPLE 5.1.6. Solve the equation

$$-8 = x-\sqrt{-2x-16}$$

SOLUTION: Isolate the radical on the left before squaring both sides:

$$\begin{aligned}
-8 &= x-\sqrt{-2x-16} \\
\sqrt{-2x-16} &= x+8
\end{aligned}$$

Square both sides now, but note that the right side is a sum of two terms, and will have to be squared as a binomial:

$$\begin{aligned}
\left(\sqrt{-2x-16}\right)^2 &= (x+8)^2 \\
-2x-16 &= x^2+16x+64
\end{aligned}$$

This is a quadratic equation, so we will move all non-zero terms to one side, factor, and solve using the zero factor property:

$$\begin{aligned}
-2x-16+2x+16 &= x^2+16x+64+2x+16 \\
0 &= x^2+18x+80 \\
0 &= (x+8)(x+10)
\end{aligned}$$

By the zero product property, this quadratic equation has solutions $x = -8$ and $x = -10$. We have to check them before concluding they are also solutions for the equation with radicals.

If $x = -8$ then the right side of the original equation is

$$\begin{aligned} x - \sqrt{-2x-16} &= (-8) - \sqrt{-2(-8)-16} \\ &= -8 - \sqrt{16-16} \\ &= -8 - \sqrt{0} \\ &= -8 \end{aligned}$$

This is equal to the left side, so -8 is a solution.

If $x = -10$ then the right side of the original equation is

$$\begin{aligned} x - \sqrt{-2x-16} &= (-10) - \sqrt{-2(-10)-16} \\ &= -10 - \sqrt{20-16} \\ &= -10 - \sqrt{4} \\ &= -10 - 2 \\ &= -12 \end{aligned}$$

This is not equal to the left side, so -10 is not a solution, although it is an *extraneous solution*.

ANSWER: $\{-8\}$

Homework 8.5.

Solve the equation.

1. $\sqrt{x} = 7$

2. $\sqrt{x} - 7 = 2$

3. $\sqrt{x} - 3 = 9$

4. $\sqrt{4 - x} = 0$

5. $\sqrt{3x + 1} = 8$

6. $\sqrt{5x - 1} = 7$

7. $4\sqrt{x} = -3$

8. $1 - \sqrt{x} = 2$

9. $\sqrt{8x + 3} = \sqrt{6x + 7}$

10. $\sqrt{7b - 9} = \sqrt{b + 3}$

11. $\sqrt{3a + 1} = a - 3$

12. $b - 7 = \sqrt{b - 5}$

13. $x - 9 = \sqrt{x - 3}$

14. $\sqrt{y + 18} = y - 2$

15. $x + 4 = 4\sqrt{x + 1}$

16. $1 + 2\sqrt{a - 1} = a$

17. $\sqrt{x^2 + 6} = x - 3$

18. $\sqrt{5x + 21} = x + 3$

19. $x = 1 + \sqrt{1 - x}$

20. $x = \sqrt{x + 5} + 7$

21. $3\sqrt{x} = \sqrt{x + 32}$

22. $2\sqrt{x - 7} = \sqrt{5x}$

23. $\sqrt{x + 5} = x - 1$

24. $\sqrt{5x + 3} = \sqrt{2x - 1}$

25. $1 + x = \sqrt{1 + 5x}$

26. $\sqrt{x + 2} - 2 = x$

Homework 8.5 Answers.

1. $\{49\}$

3. $\{144\}$

5. $\{21\}$

7. $\varnothing$

9. $\{2\}$

11. $\{8\}$

13. $\{12\}$

15. $\{0, 8\}$

17. $\varnothing$

19. $\{1\}$

21. $\{4\}$

23. $\{4\}$

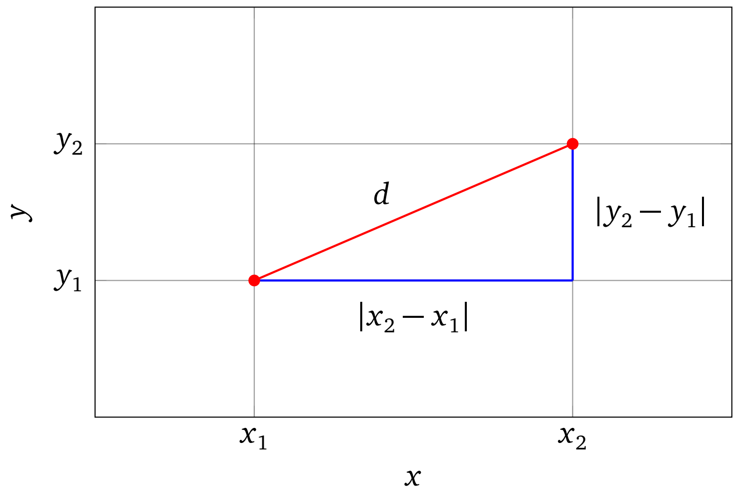
25. $\{0, 3\}$

6. Applications to Right Triangles

6.1. Distance Formula.

DEFINITION 6.1.1 (Euclidean Distance). On a coordinate plane, the *distance* d between points with coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



This definition is in full agreement with the Pythagorean theorem. Applied to the highlighted right triangle pictured above, the theorem states

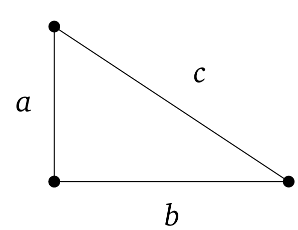
$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

If we take the principal square root of both sides of the equation, we will get our definition of the distance.

Given a right triangle with two known sides, we can use the Pythagorean theorem to solve for the length of the remaining side.

EXAMPLE 6.1.1. Given the lengths of two sides of a pictured right triangle with legs a and b and hypotenuse c, find the length of the remaining side.

$$a = 8, \quad b = 4$$



SOLUTION: We state the Pythagorean theorem for this triangle and solve for the unknown variable:

$$\begin{aligned} a^2+b^2 &= c^2 \\ 8^2+4^2 &= c^2 && \text{substitute known lengths} \\ 80 &= c^2 && \text{take principal square root of both sides} \\ \sqrt{80} &= c \end{aligned}$$

It is true that there is another solution to this quadratic equation, namely $-\sqrt{80}$, but since there are no negative lengths, the negative solution is not applicable and can be disregarded. We will state the answer with simplified radicand:

$$\begin{aligned} c &= \sqrt{80} \\ c &= \sqrt{16\cdot 5} \\ c &= \sqrt{16}\cdot\sqrt{5} \\ c &= 4\cdot\sqrt{5} \end{aligned}$$

ANSWER: $4\sqrt{5}$

EXAMPLE 6.1.2. Find the distance between the points with coordinates

$$(4,7) \quad \text{and} \quad (-5,4)$$

SOLUTION: Let $(x_1,y_1)=(4,7)$, $(x_2,y_2)=(-5,4)$, and use the distance formula:

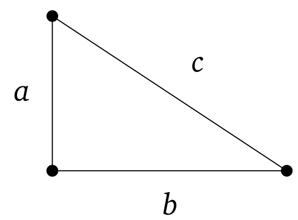
$$\begin{aligned} d &= \sqrt{(-5-4)^2+(4-7)^2} && \text{simplify sums before squaring} \\ &= \sqrt{(-9)^2+(-3)^2} \\ &= \sqrt{81+9} \\ &= \sqrt{90} \end{aligned}$$

Simplify the radicand for the answer:

$$\begin{aligned} \sqrt{90} &= \sqrt{9\cdot 10} \\ &= \sqrt{9}\cdot\sqrt{10} && \text{distributivity of } \sqrt{\ } \text{ over } \cdot \\ &= 3\cdot\sqrt{10} \end{aligned}$$

ANSWER: $3\sqrt{10}$

Homework 8.6.



Given the lengths of two sides of a pictured straight triangle with legs a and b and hypotenuse c, find the length of the remaining side.

1. $a = 13,\ b = 5$

2. $a = 9,\ c = 15$

3. $b = 1,\ c = \sqrt{10}$

4. $a = 1,\ c = \sqrt{3}$

5. $c = 10,\ b = 5\sqrt{3}$

6. $a = 24,\ b = 10$

7. $a = 18,\ c = 30$

8. $b = 1,\ c = \sqrt{2}$

9. $a = \sqrt{2},\ b = \sqrt{6}$

10. $a = 5,\ b = 5$

Find the distance between the given points.

11. $(2, 3)$ and $(6, 10)$

12. $(0, 3)$ and $(4, 0)$

13. $(-3, 2)$ and $(-1, 5)$

14. $(-2, 4)$ and $(-8, -4)$

15. $(0, -5)$ and $(-12, 0)$

16. $(2, -7)$ and $(1, -4)$

17. $(-2, -8)$ and $(6, 7)$

18. $(1, 0)$ and $(7, 3)$

Homework 8.6 Answers.

1. $c = 13$

3. $a = 3$

5. $a = 5$

7. $b = 26$

9. $c = 2\sqrt{2}$

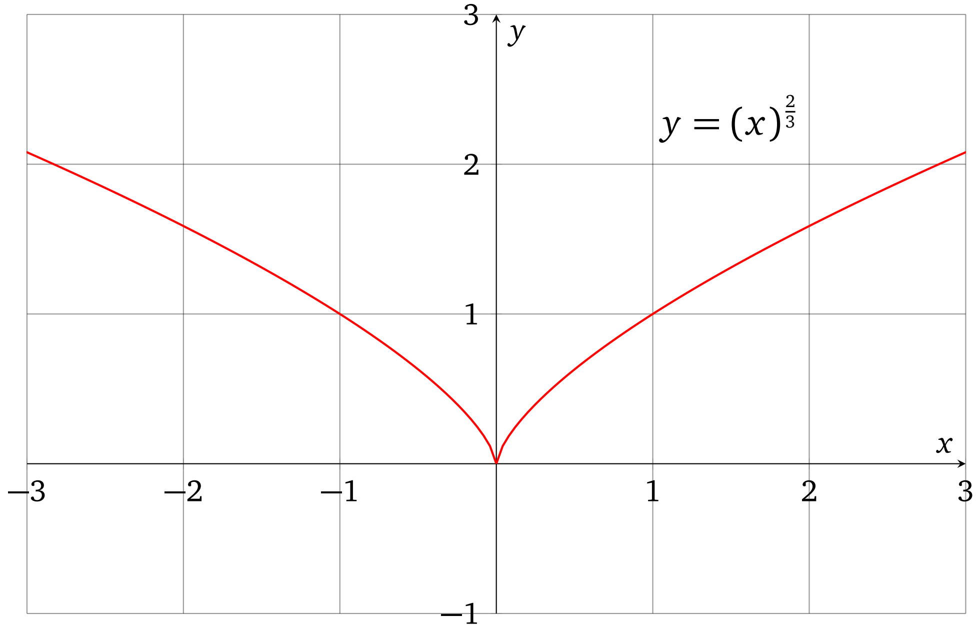
11. $\sqrt{65}$

13. $\sqrt{13}$

15. 13

17. 17

7. Rational Exponent



7.1. Rational Exponent Definition.

DEFINITION 7.1.1 (Integer Reciprocal Exponent). If b is a real number, and n is a positive integer, then the exponential expression $(b)^{1/n}$ is defined as follows:

$$(b)^{\frac{1}{n}} = y, \quad \text{where} \quad y^n = b$$

If n is even, then the positive y is chosen. Notationally, these are equivalent:

$$(b)^{\frac{1}{n}} = \sqrt[n]{b}$$

BASIC EXAMPLE 7.1.1. Some easy-to-find exponents:

$(25)^{\frac{1}{2}} = 5$ because $5^2 = 25$.

$(16)^{\frac{1}{4}} = 2$ because $2^4 = 16$.

$(-8)^{\frac{1}{3}} = -2$ because $(-2)^3 = -8$.

$(0.25)^{\frac{1}{2}} = 0.5$ because $(0.5)^2 = 0.25$.

$(-0.00001)^{\frac{1}{5}} = -0.1$ because $(-0.1)^5 = -0.00001$

EXAMPLE 7.1.1. Find $(225)^{-\frac{1}{2}}$

SOLUTION: All properties of exponent still apply, so

$$\begin{aligned}(225)^{-\frac{1}{2}} &= \left(\frac{1}{225}\right)^{\frac{1}{2}} \\ &= \frac{1}{(225)^{\frac{1}{2}}} \\ &= \frac{1}{15}\end{aligned}$$

ANSWER: $\frac{1}{15}$

DEFINITION 7.1.2 (Rational Exponent). If b is a real number, and m/n is a quotient of an integer and a positive integer in lowest terms, then the exponential expression $(b)^{\frac{m}{n}}$ is defined as follows:

$$(b)^{\frac{m}{n}} = \left((b)^{\frac{1}{n}}\right)^m$$

Note that if the base b is non-negative, then all of these are equivalent:

$$(b)^{\frac{m}{n}} = \left((b)^{\frac{1}{n}}\right)^m = ((b)^m)^{\frac{1}{n}} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

Note also that several of the integer exponent properties stop working when we allow the base b to be negative, so we are assuming throughout this section that all *variable* bases are non-negative.

THEOREM 7.1.1. Assuming that all bases are non-negative, the rational exponent obeys all the properties of the integer exponent.

Recall the properties of the integer exponent we have seen so far:

- Distributivity over multiplication:

$$(ab)^n = a^n b^n$$

- Distributivity over division:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

- The product rule:

$$b^m b^n = b^{m+n}$$

- The quotient rule:

$$\frac{b^m}{b^n} = b^{m-n}$$

- The power rule:

$$(b^m)^n = b^{mn}$$

- Negative exponent properties:

$$b^{-n} = \left(\frac{1}{b}\right)^n = \frac{1}{b^n}$$

EXAMPLE 7.1.2. Simplify and state the answer with positive exponents only:

$$\left(a^6\right)^{\frac{1}{3}}$$

SOLUTION:

$$\begin{aligned} \left(a^6\right)^{\frac{1}{3}} &= (a)^{6\cdot\frac{1}{3}} \\ &= a^2 \end{aligned}$$

ANSWER: a^2

EXAMPLE 7.1.3. Simplify and state the answer with positive exponents only:

$$\left(x^{\frac{1}{3}}y^{-1}\right)^{\frac{3}{2}} \cdot x^{\frac{5}{2}}y^2$$

SOLUTION:

$$\begin{aligned} \left(x^{\frac{1}{3}}y^{-1}\right)^{\frac{3}{2}} \cdot x^{\frac{5}{2}}y^2 &= \left(x^{\frac{1}{3}}\right)^{\frac{3}{2}}\left(y^{-1}\right)^{\frac{3}{2}} \cdot x^{\frac{5}{2}}y^2 && \text{distributivity} \\ &= x^{\frac{1}{3}\cdot\frac{3}{2}}y^{-1\cdot\frac{3}{2}} \cdot x^{\frac{5}{2}}y^2 && \text{power rule} \\ &= x^{\frac{1}{2}}y^{-\frac{3}{2}} \cdot x^{\frac{5}{2}}y^2 \\ &= x^{\frac{1}{2}+\frac{5}{2}}y^{-\frac{3}{2}+2} \\ &= x^3y^{\frac{1}{2}} \end{aligned}$$

ANSWER: $x^3y^{\frac{1}{2}}$

EXAMPLE 7.1.4. Simplify and state the answer with positive exponents only:

$$\frac{a^{\frac{13}{2}} b^0 a^{-3}}{b^{-\frac{5}{7}} a^{\frac{1}{4}}}$$

SOLUTION: We can use the product and the quotient rules add and subtract corresponding exponents:

$$\frac{a^{\frac{13}{2}} b^0 a^{-3}}{b^{-\frac{5}{7}} a^{\frac{1}{4}}} = a^{\frac{13}{2}+(-3)-\frac{1}{4}} b^{0-\left(-\frac{5}{7}\right)} \qquad \frac{13}{2} - 3 = \frac{13}{2} - \frac{6}{2} = \frac{13-6}{2} = \frac{7}{2}$$

$$= a^{\frac{7}{2}-\frac{1}{4}} b^{\frac{5}{7}} \qquad \frac{7}{2} - \frac{1}{4} = \frac{14}{4} - \frac{1}{4} = \frac{14-1}{4} = \frac{13}{4}$$

$$= a^{\frac{13}{4}} b^{\frac{5}{7}}$$

ANSWER: $a^{\frac{13}{4}} b^{\frac{5}{7}}$

Homework 8.7.

Write the expression in radical form.

1. $(2m)^{\frac{4}{5}}$
2. $(10r)^{\frac{3}{4}}$
3. $(7x)^{\frac{3}{2}}$
4. $(6b)^{\frac{4}{3}}$

Write the expression in exponential form.

5. $\sqrt[6]{v}$
6. $\sqrt{5a}$
7. $\sqrt{(4x)^5}$
8. $\sqrt[3]{c^4}$

Evaluate the expression.

9. $8^{\frac{2}{3}}$
10. $16^{\frac{1}{4}}$
11. $4^{\frac{5}{2}}$
12. $9^{\frac{3}{2}}$
13. $25^{-\frac{1}{2}}$
14. $16^{-\frac{3}{2}}$
15. $(-8)^{-\frac{4}{3}}$
16. $(-27)^{-\frac{1}{3}}$

Simplify and state the answer with positive exponents only.

17. $yx^{\frac{1}{3}} \cdot xy^{\frac{3}{2}}$
18. $4v^{\frac{5}{3}} \cdot v^{-1}$
19. $\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^{-1}$
20. $\left(x^{\frac{5}{3}}y^{-2}\right)^{0}$
21. $\dfrac{ab^3}{3ab^0}$
22. $\dfrac{2x^{\frac{1}{2}}y^{\frac{1}{3}}}{2x^{\frac{4}{3}}y^{-\frac{7}{4}}}$
23. $\left(x^{\frac{1}{2}}y^{\frac{3}{4}}\right)^{\frac{2}{3}} \cdot x^{\frac{2}{3}}y^{-\frac{5}{2}}$
24. $\left(g^3h^{-\frac{1}{3}} \cdot g^{-\frac{7}{3}}\right)^{-\frac{1}{2}}$
25. $u^2 \cdot uv\left(v^{\frac{2}{3}}\right)^3$
26. $(x \cdot xy^2)^0$
27. $\left(x^0y^{\frac{1}{3}}\right)^{\frac{3}{2}} x^0$
28. $u^{-\frac{5}{4}}v^2 \cdot \left(u^{\frac{2}{3}}\right)^{-\frac{3}{2}}$
29. $\dfrac{a^{\frac{3}{4}}b^{-1} \cdot b^{\frac{7}{4}}}{3b^{-1}}$
30. $\dfrac{2x^{-2}y^{\frac{5}{3}}}{x^{-\frac{5}{4}}y^{-\frac{5}{3}} \cdot xy^{\frac{1}{2}}}$

31. $\dfrac{ab^{\frac{1}{3}} \cdot 2b^{-\frac{5}{4}}}{4a^{-\frac{1}{2}}b^{-\frac{2}{3}}}$

32. $\dfrac{3y^{-\frac{5}{4}}}{y^{-1} \cdot 2y^{-\frac{1}{3}}}$

33. $\left(\dfrac{m^{\frac{3}{2}}n^{-2}}{m^{-1}n^{-\frac{4}{3}}}\right)^{\frac{7}{4}}$

34. $\left(\dfrac{y^{\frac{1}{3}}y^{-2}}{x^{-\frac{5}{2}}y^{-\frac{9}{2}}}\right)^{\frac{3}{2}}$

Homework 8.7 Answers.

1. $\sqrt[5]{16m^4}$

3. $\sqrt{343x^3}$

5. $v^{\frac{1}{6}}$

7. $(4x)^{\frac{5}{2}}$

9. 4

11. 32

13. $\frac{1}{5}$

15. $\frac{1}{16}$

17. $x^{\frac{4}{3}}y^{\frac{5}{2}}$

19. $\frac{1}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$

21. $\frac{b}{3}$

23. $\frac{x}{y^2}$

25. u^3v^3

27. $y^{\frac{1}{2}}$

29. $\frac{1}{3}b^{\frac{7}{4}}a^{\frac{3}{4}}$

31. $\frac{a^{\frac{3}{2}}}{2b^{\frac{1}{4}}}$

33. $\frac{m^{\frac{35}{8}}}{n^{\frac{7}{6}}}$

Practice Test 8

Simplify each expression, assuming that all variable bases and radicands are non-negative:

1. $\sqrt{121}$

2. $\sqrt{\dfrac{4}{9}}$

3. $\sqrt[3]{-8}$

4. $\sqrt{x^{2018}}$

5. $(\sqrt{x-y})^2$

6. $\sqrt[4]{81}$

7. $\sqrt[5]{x^{15}}$

8. $\sqrt{72x^5}$

9. $\sqrt{3y^7} \cdot \sqrt{27y^3}$

10. $\dfrac{\sqrt{7x^7}}{\sqrt{28x}}$

11. $\sqrt{\dfrac{54x}{3y}}$

12. $\sqrt{12}(3\sqrt{2} - \sqrt{8})$

Rationalize the denominator and simplify:

13. $\dfrac{6}{3+\sqrt{7}}$

14. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{6}-\sqrt{5}}$

Solve each equation:

15. $\sqrt{2x+1} = \sqrt{x+2}$

16. $\sqrt{5x+1} - 6 = 0$

17. $x - 2 = \sqrt{2x-4}$

18. $\sqrt{2x-5} + 4 = x$

---

19. Find the length of the leg b of a straight triangle, if the other leg a is 6 cm, and the hypotenuse c is 14 cm.

20. Find the distance between points $(-5,-9)$ and $5,-3$.

Evaluate the expression:

21. $100^{\frac{3}{2}}$

22. $(-64)^{-\frac{2}{3}}$

Simplify the expression and state the answer with positive exponents only:

23. $\left(x^{\frac{1}{3}}y^{-\frac{1}{2}}\right)^2 \cdot x^{-\frac{5}{3}}y^{-2}$

24. $\dfrac{a^{\frac{1}{2}}b^{-\frac{1}{5}}}{a^{\frac{1}{3}}b^{-\frac{11}{5}}}$

Practice Test 8 Answers.

1. 11

2. $\frac{2}{3}$

3. -2

4. x^{1009}

5. $x-y$

6. 3

7. x^3

8. $6x^2\sqrt{2x}$

9. $9y^5$

10. $x^3/2$

11. $\frac{3\sqrt{2xy}}{y}$

12. $2\sqrt{6}$

13. $9-3\sqrt{7}$

14. $\sqrt{10}+2\sqrt{3}+\sqrt{15}+3\sqrt{2}$

15. $\{1\}$

16. $\{7\}$

17. $\{2,4\}$

18. $\{7\}$

19. $4\sqrt{10}$

20. $2\sqrt{34}$

21. 1000

22. $1/16$

23. $\frac{1}{xy^3}$

24. $a^{\frac{1}{6}}b^2$

CHAPTER 9

Quadratic Equations

1. Complex Numbers

1.1. Imaginary Unit. Complex numbers were initially designed to assist in factoring polynomials around 1545 by Gerolamo Cardano. It was observed that an equation such as

$$x^2 = -1$$

which is provably without a real solution, can be postulated to have a solution which is not a real number, and nothing goes wrong as a result. In fact, one could say the situation improves somewhat. It becomes possible to construct a set of *complex numbers*, which includes all real numbers as well as all numbers which happen to be solutions of polynomial equations with real or complex coefficients. All of the real number axioms hold true for the complex numbers, so most theorems presented in this text hold for the complex numbers almost word for word.

DEFINITION 1.1.1 (Imaginary Unit). The solutions of the equation

$$x^2 = -1$$

are complex numbers i and its opposite $-i$:

$$i^2 = (-i)^2 = -1$$

i is usually called the *imaginary unit*, and it happens to be the principal square root of the negative unit:

$$\sqrt{-1} = i$$

THEOREM 1.1.1. If x is a positive real number, then the principal square root of $-x$ is

$$\sqrt{-x} = \sqrt{x} \cdot i$$

EXAMPLE 1.1.1. Simplify the expression $\sqrt{-100}$

SOLUTION:

$$\sqrt{-100} = \sqrt{100} \cdot i = 10i$$

ANSWER: $10i$

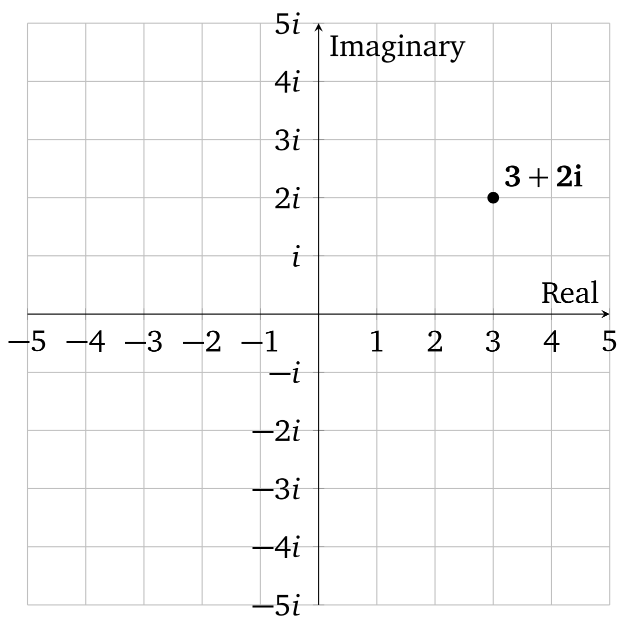
EXAMPLE 1.1.2. Simplify the expression $\sqrt{-20}$

SOLUTION:

$$\sqrt{-20} = \sqrt{20} \cdot i = 2\sqrt{5} \cdot i$$

ANSWER: $2\sqrt{5} \cdot i$

1.2. The Set of Complex Numbers. Just like the set of real numbers can be visualized as a number line, the set of complex numbers can be visualized as a number plane.



This plane works a lot like the Cartesian plane, but the coordinates are interpreted in a particular way. Every point on this plane represents a distinct complex number. The points on the real axis are the familiar real numbers. The imaginary unit i is located one unit above the origin, and the points on the imaginary axis are the real multiples of i. Every other point with coordinates (a, b), where a and b are real numbers, is a complex number which can be written as $a + bi$.

DEFINITION 1.2.1. A *complex number* is any number z which can be written in the form

$$a + bi$$

where a is a real number called the *real part of* z, b is a real number called the *imaginary part of* z, and i is the imaginary unit.

1.3. Adding and Multiplying Complex Numbers.

EXAMPLE 1.3.1. Simplify the expression $(3+5i)-(2+7i)$

SOLUTION: We treat i as if it was a variable, and combine the like terms:

$$\begin{aligned} (3+5i)-(2+7i) &= 3+5i-2-7i && \text{removed parentheses} \\ &= 1-2i && \text{combined like terms} \end{aligned}$$

ANSWER: $1-2i$

EXAMPLE 1.3.2. Simplify the expression $(1+6i)(2-i)$

SOLUTION: We treat i as if it was a variable, multiply the binomials, and combine the like terms:

$$\begin{aligned} (1+6i)(2-i) &= 2-1i+12i-6i^2 \\ &= 2+11i-6i^2 && i^2 \text{ can be replaced by } -1 \\ &= 2+11i-6(-1) \\ &= 2+11i+6 \\ &= 8+11i && \text{combined like terms} \end{aligned}$$

ANSWER: $8+11i$

EXAMPLE 1.3.3. Simplify the expression $\dfrac{8+\sqrt{-16}}{4}$

SOLUTION: We simplify the radicand, and then cancel common factors:

$$\begin{aligned} \frac{8+\sqrt{-16}}{4} &= \frac{8+4i}{4} && \text{rewrite in terms of } i \\ &= \frac{4(2+i)}{4} && \text{factored out GCF} \\ &= 2+i && \text{canceled common factor 4} \end{aligned}$$

ANSWER: $2+i$

EXAMPLE 1.3.4. Simplify the expression $\dfrac{20-\sqrt{-50}}{10}$

SOLUTION:

$$
\begin{aligned}
\frac{20-\sqrt{-50}}{10} &= \frac{20-\sqrt{50}\cdot i}{10} && \text{rewrite in terms of } i \\
&= \frac{20-\sqrt{25\cdot 2}\cdot i}{10} && \text{simplify the radicand} \\
&= \frac{20-\sqrt{25}\sqrt{2}\cdot i}{10} && \text{distributivity of } \sqrt{} \text{ over } \cdot \\
&= \frac{20-5\sqrt{2}\cdot i}{10} && \sqrt{25}=5 \\
&= \frac{5(4-\sqrt{2}\cdot i)}{10} && \text{factored out GCF} \\
&= \frac{4-\sqrt{2}\cdot i}{2} && \text{canceled common factor 5}
\end{aligned}
$$

If desired, this expression could be stated in the form $a+bi$ by rewriting the fraction as a sum:

$$
\begin{aligned}
\frac{4-\sqrt{2}\cdot i}{2} &= \frac{4}{2}-\frac{\sqrt{2}}{2}i \\
&= 2-\frac{\sqrt{2}}{2}i
\end{aligned}
$$

But we will be content with simplified fractions for answers:

ANSWER: $\dfrac{4-\sqrt{2}\cdot i}{2}$

Homework 9.1.

Simplify the given expression and state the answer in the form bi, where b is real.

1. $\sqrt{-5}$

2. $\sqrt{-6}$

3. $\sqrt{-49}$

4. $\sqrt{-81}$

5. $\sqrt{-45}$

6. $\sqrt{-18}$

7. $\sqrt{-120}$

8. $\sqrt{-160}$

Simplify the given expression.

9. $(7+3i)+(14+2i)$

10. $(12-4i)+(-1+6i)$

11. $(21-8i)-(5+5i)$

12. $(-10-i)-(2-4i)$

13. $(2-3i)-(9-3i)$

14. $(6-11i)+(17i-6)$

15. $(1+i)(1-2i)$

16. $(3-i)(4+i)$

17. $(10-3i)(2i)$

18. $(-5i)(2+4i)$

19. $30(-2+0.5i)$

20. $\frac{2}{3}(24-30i)$

21. $(7+5i)(7-5i)$

22. $(3-6i)(3+6i)$

23. $(4+i)^2$

24. $(1-3i)^2$

25. $\dfrac{-6+\sqrt{-4}}{3}$

26. $\dfrac{15-\sqrt{-9}}{5}$

27. $\dfrac{18+\sqrt{-8}}{-2}$

28. $\dfrac{-10+\sqrt{-25}}{-10}$

29. $\dfrac{-24-\sqrt{-27}}{-6}$

30. $\dfrac{-4+\sqrt{-28}}{6}$

Homework 9.1 Answers.

1. $\sqrt{5} \cdot i$

3. $7i$

5. $3\sqrt{5} \cdot i$

7. $2\sqrt{30} \cdot i$

9. $21 + 5i$

11. $16 - 13i$

13. -7

15. $3 - 2i$

17. $6 + 20i$

19. $-60 + 15i$

21. 24

23. $15 + 8i$

25. $-2 + \frac{2}{3}i$

27. $-9 - \sqrt{2} \cdot i$

29. $4 + \frac{\sqrt{3}}{2}i$

2. Square Root Property

2.1. Square Root Property.

THEOREM 2.1.1. For every non-zero real or complex number z there exist precisely two distinct complex numbers w such that $w^2 = z$. One of the numbers is the principal square root of z, $\sqrt{z}$, and the other number is its opposite $-\sqrt{z}$.

BASIC EXAMPLE 2.1.1. If $w^2 = 25$, then $w = 5$ or $w = -5$. In the case when the square is positive, both solutions are real.

If $w^2 = -4$, then $w = 2i$ or $w = -2i$:

$$\begin{aligned} (2i)^2 &= 2^2 i^2 \\ &= 4(-1) \\ &= -4 \end{aligned}$$

$-2i$ can be verified as a solution as well. In this case there are no real solutions, but there are two distinct complex solutions.

THEOREM 2.1.2 (Square Root Property). Some equations can be solved by taking square roots of both sides. If X is an algebraic expression and r is a real number, then solving the equation

$$X^2 = r$$

amounts to combining the solution sets of equations

$$X = \sqrt{r}$$

and

$$X = -\sqrt{r}$$

EXAMPLE 2.1.1. Solve the equation

$$x^2 = 9$$

SOLUTION: Take square roots of both sides. Either

$$x = \sqrt{9} = 3$$

or

$$x = -\sqrt{9} = -3$$

ANSWER: $\{-3, 3\}$

EXAMPLE 2.1.2. Solve the equation

$$y^2 = 12$$

SOLUTION: Take square roots of both sides. Either

$$y = \sqrt{12} = 2\sqrt{3}$$

or

$$y = -\sqrt{12} = -2\sqrt{3}$$

We can state this solution set using the roster notation $\left\{-2\sqrt{3}, 2\sqrt{3}\right\}$, but it is more traditional to use the plus-minus notation when the quadratic solutions are irrational and/or complex.

ANSWER: $\pm 2\sqrt{3}$

DEFINITION 2.1.1. Many quadratic equations have irrational solution sets of the form

$$\left\{a + \sqrt{b},\ a - \sqrt{b}\right\}$$

and many others have complex solutions sets of the form

$$\{a + bi,\ a - bi\}$$

In cases like that we will use a *plus-minus notation* in order to be more concise:

$$\begin{aligned} \left\{a + \sqrt{b},\ a - \sqrt{b}\right\} &= a \pm \sqrt{b} \\ \{a + bi,\ a - bi\} &= a \pm bi \end{aligned}$$

BASIC EXAMPLE 2.1.2. Some solution sets expressed two ways:

roster notation	**plus-minus notation**
$\left\{2 - 3\sqrt{5},\ 2 + 3\sqrt{5}\right\}$	$2 \pm 3\sqrt{5}$
$\{-6i,\ 6i\}$	$\pm 6i$
$\left\{7 - 2\sqrt{3} \cdot i,\ 7 + 2\sqrt{3} \cdot i\right\}$	$7 \pm 2\sqrt{3} \cdot i$
$\left\{\frac{8 - 9i}{5},\ \frac{8 + 9i}{5}\right\}$	$\frac{8 \pm 9i}{5}$

2.2. Solving Equations $(nx+k)^2 = r$.

EXAMPLE 2.2.1. Solve the equation $(x+1)^2 = 25$.

SOLUTION: By the square root property, either

$$\begin{aligned} \sqrt{(x+1)^2} &= \sqrt{25} \\ x+1 &= \sqrt{25} \\ x+1 &= 5 \\ x &= 4 \end{aligned}$$

or

$$\begin{aligned} \sqrt{(x+1)^2} &= -\sqrt{25} \\ x+1 &= -\sqrt{25} \\ x+1 &= -5 \\ x &= -6 \end{aligned}$$

The solutions are rational, so we state the solution set in roster notation:

ANSWER: $\{-6, 4\}$

EXAMPLE 2.2.2. Solve the equation $(t+2)^2 = 5$

SOLUTION: By the square root property, either

$$\begin{aligned} t+2 &= \sqrt{5} \\ t &= -2+\sqrt{5} \end{aligned}$$

or

$$\begin{aligned} t+2 &= -\sqrt{5} \\ t &= -2-\sqrt{5} \end{aligned}$$

When the solution set for a quadratic equation is irrational or complex, it is traditional to state the answer

$$\{-2-\sqrt{5}, \; -2+\sqrt{5}\}$$

in the plus-minus form $-2 \pm \sqrt{5}$

ANSWER: $-2 \pm \sqrt{5}$

EXAMPLE 2.2.3. Solve the equation $(2y-3)^2 = 9$

SOLUTION: By the square root property, either

$$\begin{aligned} \sqrt{(2y-3)^2} &= \sqrt{9} \\ 2y-3 &= \sqrt{9} \\ 2y-3 &= 3 \\ 2y &= 6 && \text{added 3 to both sides} \\ y &= 3 && \text{divided both sides by 2} \end{aligned}$$

or

$$\begin{aligned} \sqrt{(2y-3)^2} &= -\sqrt{9} \\ 2y-3 &= -\sqrt{9} \\ 2y-3 &= -3 \\ 2y &= 0 \\ y &= 0 \end{aligned}$$

The solutions are rational, so we state the solution set in roster notation:

ANSWER: $\{0,3\}$

2.3. Complex Solutions.

EXAMPLE 2.3.1. Solve the equation $z^2 = -16$

SOLUTION: By the square root property, either

$$\begin{aligned} z &= \sqrt{-16} \\ &= \sqrt{16} \cdot i \\ &= 4i \end{aligned}$$

or

$$\begin{aligned} z &= -\sqrt{-16} \\ &= -4i \end{aligned}$$

ANSWER: $\pm 4i$

EXAMPLE 2.3.2. Solve the equation $(x-5)^2 = -100$

SOLUTION: By the square root property, either

$$\begin{aligned} x-5 &= \sqrt{-100} \\ x-5 &= 10i \\ x &= 5+10i \end{aligned}$$

or

$$\begin{aligned} x-5 &= -\sqrt{-100} \\ x-5 &= -10i \\ x &= 5-10i \end{aligned}$$

ANSWER: $5 \pm 10i$

Homework 9.2.

Solve the equation by finding all real solutions.

1. $x^2 = 36$

2. $y^2 = 144$

3. $(x-6)^2 = 4$

4. $(x+7)^2 = 16$

5. $(4n-5)^2 = 1$

6. $(5n-1)^2 = 49$

7. $(x-3)^2 = 28$

8. $(x+9)^2 = 40$

9. $(-3z+4)^2 = 0$

10. $(5z+8)^2 = 0$

11. $\left(x+\frac{1}{2}\right)^2 = \frac{9}{4}$

12. $\left(x-\frac{3}{4}\right)^2 = \frac{1}{16}$

Solve the equation by finding all complex solutions.

13. $z^2 = -2$

14. $z^2 = -10$

15. $(y+2)^2 = -25$

16. $(y-4)^2 = -36$

17. $(z-4)^2 = -9$

18. $(2z+5)^2 = -49$

19. $(x+4)^2 = -75$

20. $(y-5)^2 = -32$

Homework 9.2 Answers.

1. $\{-6, 6\}$

3. $\{4, 8\}$

5. $\left\{\frac{3}{2}, 1\right\}$

7. $3 \pm 2\sqrt{7}$

9. $\left\{\frac{4}{3}\right\}$

11. $\{-2, 1\}$

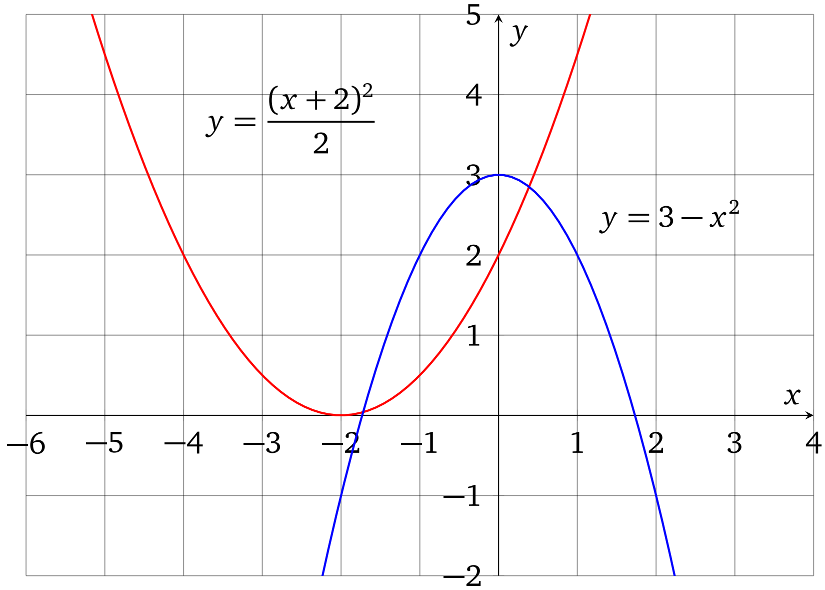
13. $\pm\sqrt{2} \cdot i$

15. $-2 \pm 5i$

17. $4 \pm 3i$

19. $-3 \pm 5\sqrt{3} \cdot i$

3. Completing the Square



3.1. Completing the Square for $x^2 + bx$. Our goal for this chapter is to be able to solve every quadratic equation, and we will be able to do so because every quadratic equation can be put in a form $(x+v)^2 = r$. The crucial step in that transformation is rewriting an expression of the type $x^2 + bx$ as a sum of a binomial square and a constant:

$$x^2 + bx = (x+v)^2 - t$$

As it happens, there is an easy way to figure out v and t from any given b.

THEOREM 3.1.1 (Completing the Square). For any real or complex number b, the polynomial $x^2 + bx$ has an equivalent form

$$x^2 + bx = (x+v)^2 - t$$

where $v = \frac{b}{2}$ and $t = \left(\frac{b}{2}\right)^2$

PROOF.

$$\begin{aligned}(x+v)^2 - t &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \\ &= x^2 + bx\end{aligned}$$

□

EXAMPLE 3.1.1. Complete the square for the expression $x^2 + 6x$

SOLUTION: Using the formula from the theorem 3.1.1, $b = 6$, so in order to complete the square we need to add and subtract t, which we compute as follows:

$$t = \left(\frac{b}{2}\right)^2 = \left(\frac{6}{2}\right)^2 = 3^2 = 9$$

and so

$$\begin{aligned} x^2 + 6x &= x^2 + 6x + 9 - 9 \\ &= (x+3)^2 - 9 \end{aligned}$$

ANSWER: $(x+3)^2 - 9$

EXAMPLE 3.1.2. Complete the square for the expression $x^2 - 14x$

SOLUTION: Using the formula from the theorem 3.1.1, $b = -14$, so in order to complete the square we need to add and subtract t, which we compute as follows:

$$t = \left(\frac{b}{2}\right)^2 = \left(\frac{-14}{2}\right)^2 = (-7)^2 = 49$$

and so

$$\begin{aligned} x^2 - 14x &= x^2 - 14x + 49 - 49 \\ &= (x-7)^2 - 49 \end{aligned}$$

ANSWER: $(x-7)^2 - 49$

EXAMPLE 3.1.3. Complete the square for the expression $x^2 + 5x$

SOLUTION: Using the formula from the theorem 3.1.1, $b = 5$, so in order to complete the square we need to add and subtract t, which we compute as follows:

$$t = \left(\frac{b}{2}\right)^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

and so

$$\begin{aligned} x^2+5x &= x^2+5x+\frac{25}{4}-\frac{25}{4} \\ &= \left(x+\frac{5}{2}\right)^2-\frac{25}{4} \end{aligned}$$

ANSWER: $\left(x+\frac{5}{2}\right)^2-\frac{25}{4}$

3.2. Solving Quadratic Equations by Completing the Square.

EXAMPLE 3.2.1. Solve the equation $x^2-8x+2=0$

SOLUTION: Complete the square for the binomial x^2-8 first. Since we are dealing with an equation, we can add the constant required to complete the square to both sides. In this case, the coefficient for the linear term is −8, and so the constant to complete the square is

$$\left(\frac{-8}{2}\right)^2=(-4)^2=16$$

$$\begin{aligned} x^2-8x+2 &= 0 && \text{constant 2 belongs on the right} \\ x^2-8x+2-2 &= 0-2 && \\ x^2-8x &= -2 && \\ x^2-8x+16 &= -2+16 && \text{add 16 to both sides to complete the square} \\ (x-4)^2 &= 14 && \text{complete the square and simplify the right side} \end{aligned}$$

Now we apply the square root property to get the answers. Either

$$\begin{aligned} \sqrt{(x-4)^2} &= \sqrt{14} \\ x-4 &= \sqrt{14} \\ x &= 4+\sqrt{14} \end{aligned}$$

or

$$\begin{aligned} \sqrt{(x-4)^2} &= -\sqrt{14} \\ x-4 &= -\sqrt{14} \\ x &= 4-\sqrt{14} \end{aligned}$$

ANSWER: $4\pm\sqrt{14}$

EXAMPLE 3.2.2. Solve the equation $x^2 + 6x + 10 = 0$

SOLUTION: Complete the square for the binomial $x^2 + 6x$ first. Since we are dealing with an equation, we will add the constant required to complete the square to both sides. In our case, the coefficient for the linear term is 6, so the constant to complete the square is

$$\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$$

$$\begin{aligned} x^2 + 6x + 10 &= 0 && \text{the constant 10 belongs on the right} \\ x^2 + 6x + 10 - 10 &= 0 - 10 \\ x^2 + 6x &= -10 \\ x^2 + 6x + 9 &= -10 + 9 && \text{add 9 to both sides to complete the square} \\ (x+3)^2 &= -1 \end{aligned}$$

Now we apply the square root property to get the answers. Either

$$\begin{aligned} x + 3 &= \sqrt{-1} \\ x + 3 &= i \\ x &= -3 + i \end{aligned}$$

or

$$\begin{aligned} x + 3 &= -\sqrt{-1} \\ x + 3 &= -i \\ x &= -3 - i \end{aligned}$$

As always in an equation, we can check each answer by comparing the values of the two sides of our equation. If $x = -3 + i$, then

$$\begin{aligned} x^2 + 6x + 10 &= (-3+i)^2 + 6(-3+i) + 10 \\ &= 9 - 6i + i^2 - 18 + 6i + 10 && \text{distributed} \\ &= 1 + i^2 && \text{combined like terms} \\ &= 0 && i^2 = -1 \end{aligned}$$

If $x = -3 - i$, then

$$\begin{aligned} x^2 + 6x + 10 &= (-3-i)^2 + 6(-3-i) + 10 \\ &= 9 + 6i + i^2 - 18 - 6i + 10 && \text{distributed} \\ &= 1 + i^2 && \text{combined like terms} \\ &= 0 && i^2 = -1 \end{aligned}$$

ANSWER: $-3 \pm i$

EXAMPLE 3.2.3. Solve the equation $3x^2 - x - 6 = 0$

SOLUTION: Before we can complete the square, we need to divide both sides by the leading coefficient 3:

$$x^2 - \frac{1}{3}x - 2 = 0$$

Now we complete the square by adding

$$\left(-\frac{1}{3} \div 2\right)^2 = \left(-\frac{1}{6}\right)^2 = \frac{1}{36}$$

to both sides.

$$\begin{aligned} x^2 - \frac{1}{3}x &= 2 && \text{constant belongs on the right} \\ x^2 - \frac{1}{3}x + \frac{1}{36} &= 2 + \frac{1}{36} \\ \left(x - \frac{1}{6}\right)^2 &= \frac{72}{36} + \frac{1}{36} && \text{found LCD on the right} \\ \left(x - \frac{1}{6}\right)^2 &= \frac{73}{36} \end{aligned}$$

Now we can apply the square root property. This time we will introduce the plus-minus notation right into the equation, saving ourselves some time:

$$\begin{aligned} \sqrt{\left(x - \frac{1}{6}\right)^2} &= \pm\sqrt{\frac{73}{36}} \\ x - \frac{1}{6} &= \pm\sqrt{\frac{73}{36}} \\ x - \frac{1}{6} &= \pm\frac{\sqrt{73}}{6} \\ x &= \frac{1}{6} \pm \frac{\sqrt{73}}{6} \\ x &= \frac{1 \pm \sqrt{73}}{6} \end{aligned}$$

ANSWER: $\dfrac{1 \pm \sqrt{73}}{6}$

Homework 9.3.

Complete the square for the given expression.

1. $x^2 + 8x$

2. $x^2 - 10x$

3. $x^2 - 20x$

4. $x^2 + 18x$

5. $x^2 + 13x$

6. $x^2 - 3x$

Solve the given equation by completing the square.

7. $x^2 + 2x - 4 = 0$

8. $x^2 + 4x - 5 = 0$

9. $x^2 - 24x + 21 = 0$

10. $a^2 - 10a + 20 = 0$

11. $b^2 - 8b + 16 = 0$

12. $c^2 + 10c + 25 = 0$

13. $a^2 - 3a - 10 = 0$

14. $x^2 - 3x - 4 = 0$

15. $5x^2 + 6x + 1 = 0$

16. $2b^2 - 7b - 10 = 0$

Find complex solutions for the equation by completing the square.

17. $x^2 - 6x + 25 = 0$

18. $x^2 + 2x + 5 = 0$

Homework 9.3 Answers.

1. $(x+4)^2-16$

3. $(x-10)^2-100$

5. $(x+6.5)^2-42.25$

7. $-1\pm\sqrt{5}$

9. $12\pm\sqrt{123}$

11. $\{4\}$

13. $\{-2,5\}$

15. $\{-0.2,-1\}$

17. $3\pm4i$

4. Quadratic Formula

THEOREM 4.1.1 (Quadratic Formula). Let a, b, and c be real numbers, $a \neq 0$, and consider the quadratic equation in one variable x in standard form:

$$ax^2 + bx + c = 0$$

where a is the coefficient of the quadratic term, b is the coefficient of the linear term, c is the constant term. Then the solutions of the equation can be computed with

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Moreover, the radicand $b^2 - 4ac$ is known as the *discriminant*. If the discriminant is positive, then there are two distinct real solutions; if $b^2 - 4ac = 0$, then there is a unique real solution; and if $b^2 - 4ac < 0$, then there are two distinct complex solutions.

PROOF. Deriving the formula amounts to solving this equation by completing the square:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c \qquad \text{moved the constant to the right}$$

$$x^2 + \frac{bx}{a} = \frac{-c}{a} \qquad \text{divided both sides by the leading coefficient } a$$

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \qquad \text{completed the square with } \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \qquad \text{found common denominator on the right}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{subtracted fractions on the right}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{applied the square root property}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \qquad \text{distributed } \sqrt{\ } \text{ over the fraction}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{simplified denominator}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{isolated } x \text{ on the left}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{quadratic formula in plus-minus notation}$$

□

EXAMPLE 4.1.1. Solve the equation $x^2 - 3x + 2 = 0$.

SOLUTION: $a = 1$, $b = -3$, and $c = 2$, so the discriminant

$$b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot 2 = 9 - 8 = 1,$$

and hence the solutions are

$$x_1 = \frac{-(-3) + \sqrt{1}}{2 \cdot 1} = 2$$

$$x_2 = \frac{-(-3) - \sqrt{1}}{2 \cdot 1} = 1$$

Just like for all equations, it is very easy to check the answer. If x_1 and x_2 are the solutions, then substituting them for x in the original equation should produce identities:

$$\begin{aligned} 2^2 - 3 \cdot 2 + 2 &= 0 \\ 4 - 6 + 2 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} 1^2 - 3 \cdot 1 + 2 &= 0 \\ 1 - 3 + 2 &= 0 \\ 0 &= 0 \end{aligned}$$

ANSWER: $\{1, 2\}$

EXAMPLE 4.1.2. Solve the equation $4x^2 - 1 = 0$.

SOLUTION: $a = 4$, $b = 0$, and $c = -1$, so the discriminant

$$b^2 - 4ac = 0^2 - 4 \cdot 4 \cdot (-1) = 16$$

and hence the solutions are

$$x_1 = \frac{-0 + \sqrt{16}}{2 \cdot 4} = \frac{4}{8} = \frac{1}{2}$$

$$x_2 = \frac{-0 - \sqrt{16}}{2 \cdot 4} = \frac{-4}{8} = -\frac{1}{2}$$

ANSWER: $\{-1/2, 1/2\}$

EXAMPLE 4.1.3. Solve the equation

$$x(x+8) = 2x - 9$$

SOLUTION: First we have to bring the equation into the standard form:

$$\begin{aligned} x(x+8) &= 2x-9 \\ x^2+8x &= 2x-9 \\ x^2+8x-2x+9 &= 0 \\ x^2+6x+9 &= 0 \end{aligned}$$

$a = 1$, $b = 6$, and $c = 9$, and so the discriminant

$$b^2 - 4ac = 36 - 4 \cdot 1 \cdot 9 = 36 - 36 = 0$$

and the only solution is

$$x_1 = x_2 = \frac{-6 \pm \sqrt{0}}{2 \cdot 1} = -3$$

ANSWER: $\{-3\}$

4.2. Complex Solutions.

EXAMPLE 4.2.1. Solve the equation

$$x^2 - 2x + 2 = 0$$

SOLUTION: $a = 1$, $b = -2$, and $c = 2$, so the discriminant

$$b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot (2) = 4 - 8 = -4$$

Since -4 is negative, there are no real solutions, but we can still apply the quadratic formula to obtain the complex solutions:

$$x_1 = \frac{-(-2) \pm \sqrt{-4}}{2 \cdot 1} = \frac{2 \pm 2i}{2} = 1 \pm i$$

ANSWER: $1 \pm i$

Homework 9.4.

Use the quadratic formula to find all real solutions.

1. $x^2 - 5x + 6 = 0$

2. $x^2 + 3x - 40 = 0$

3. $x^2 - 2x - 3 = 0$

4. $y^2 - 8y + 20 = 0$

5. $x^2 - 7x = 0$

6. $6x^2 = -3x$

7. $x^2 - 9 = 0$

8. $5y^2 = \frac{1}{5}$

9. $3x^2 - 7x + 2 = 0$

10. $3x^2 + 7x + 4 = 0$

11. $x^2 + 8x + 16 = 0$

12. $y^2 - 10y = -25$

13. $-9x^2 + 6x + 8 = 0$

14. $2x(x + 3) = 5(x + 2)$

15. $x^2 - 4x + 7 = 0$

16. $x^2 + 6x - 2 = 0$

Use the quadratic formula to find complex solutions for each equation.

17. $x^2 + 1 = 0$

18. $-3x^2 = 75$

19. $x^2 - 6x + 13 = 0$

20. $x^2 + 4x + 8 = 0$

21. $z^2 + 2z + 3 = 0$

22. $-x^2 + 7x - 12 = 0$

Homework 9.4 Answers.

1. $\{2, 3\}$

3. $\{-1, 3\}$

5. $\{0, 7\}$

7. $\{-3, 3\}$

9. $\left\{\frac{1}{3}, 2\right\}$

11. $\{4\}$

13. $\left\{-\frac{2}{3}, \frac{4}{3}\right\}$

15. $2 \pm \sqrt{3} \cdot i$

17. $\pm i$

19. $-3 \pm 2i$

21. $-1 \pm \sqrt{2} \cdot i$

Practice Final

1. Solve the formula for y:

$$4y + 3x = 5 - y$$

2. Solve the inequality

$$2(x-3) \geq 5x + 12$$

3. Frodo's journey to Mount Doom was three times longer than his journey back home, and the whole trip, there and back again, took 240 days. Find the duration of the journey to the Mount Doom and the duration of the homeward journey.

4. One week Yasmin worked 10 hours waiting tables and 3 hours editing a book, and she made \$195. Next week she waited tables for 5 hours and spent 5 hours editing a book, and she made \$185. Find the hourly wage for each job.

5. Find an equation of the line with slope $-\frac{1}{3}$ and y-intercept $(0,-2)$. State the answer in the slope-intercept form.

6. Find an equation of the line with slope -2 that passes through the point $(-1,3)$. State the answer in the slope-intercept form.

7. Find the slope and an equation of the line that passes through the points $(1,-5)$ and $(3,-1)$. State the answer in the point-slope form.

8. Solve the system of linear equations by graphing:

$$\begin{cases} y + x &= 3 \\ 2y - x &= -6 \end{cases}$$

9. Solve the system:

$$\begin{cases} 17x + 20y &= 26 \\ -17x + 10y &= 64 \end{cases}$$

10. Solve the system:

$$\begin{cases} 2(x-1) &= 6y \\ 3y - x + 2 &= 0 \end{cases}$$

Simplify the expression and state the answer in scientific notation:

11. $\dfrac{42 \times 10^{-2}}{5 \times 10^{-4}}$

12. $(5.5 \times 10^{3})(3.4 \times 10^{-5})$

Simplify the expression and state the answer with positive exponents only:

13. $\left(\dfrac{x^3 y^{-2}}{x^{-1} y^{-4}}\right)^{-2}$

14. $(-3x^{-5}y^{7})(-5x^{0}y^{-4})$

Factor the expression completely:

15. $z^4 - 81y^4$

16. $8x - 4 + 6x^3 - 3x^2$

Solve the equation by factoring:

17.
$$6x^2 = -10x$$

18.
$$100 - 4x^2 = 0$$

19.
$$x^3 - 2x^2 = 24x$$

Simplify the expression:

20. $\sqrt{100x^2}$

21. $\sqrt[3]{x^{24}}$

22. $\sqrt{45x^2}$

23. $\sqrt{242x^6y^7}$

24. $\dfrac{\sqrt{8x^7}}{\sqrt{2xy^2}}$

Solve the equation:

25.
$$\frac{7}{x+3} = \frac{x}{4}$$

26.
$$\sqrt{x-3} + 6 = 4$$

27.
$$5\sqrt{x} = \sqrt{x} + 2$$

28.
$$\sqrt{2x+5} - 1 = x$$

---

29. Solve the equation by taking square roots of both sides:
$$(x+3)^2 = 20$$

30. Solve the equation by completing the square:
$$x^2 + 18x + 1 = 0$$

Solve the equation by using the quadratic formula:

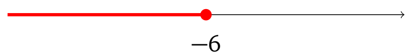
31.
$$x^2 - 4x + 1 = 0$$

32.
$$4x^2 - 4x - 35 = 0$$

Practice Final Answers.

1. $y = \dfrac{5-3x}{5}$

2. $(-\infty, -6]$



3.
x and y are the durations for going there and back again respectively, in days

$$\begin{cases} x = 3y \\ x + y = 240 \end{cases}$$

solution: $x = 180, \quad y = 60$

4.
x and y are the wages for waiting and editing respectively, in dollars per day

$$\begin{cases} 10x + 3y = 195 \\ 5x + 5y = 185 \end{cases}$$

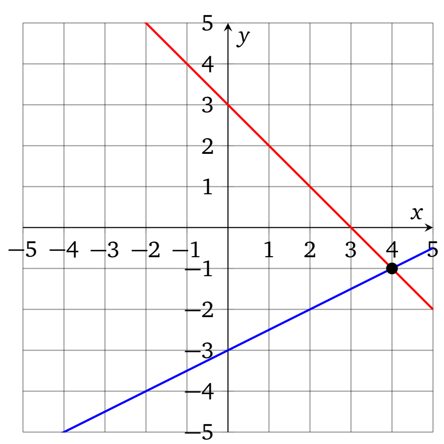
solution: $x = 12, \quad y = 25$

5. $y = -\dfrac{1}{3}x - 2$

6. $y = -2x + 1$

7. $y + 5 = 2(x - 1)$

8. $(4, -1)$



9. $(-2, 3)$

10. no solution

11. 8.4×10^2

12. 1.87×10^{-1}

13. $\dfrac{1}{y^4 x^8}$

14. $\dfrac{15y^3}{x^5}$

15. $(z + 3y)(z - 3y)(z^2 + 9y^2)$

16. $(4 + 3x^2)(2x - 1)$

17. $\{0, -5/3\}$

18. $\{-5, 5\}$

19. $\{0, -4, 6\}$

20. $10x$

21. x^8

22. $3x\sqrt{5}$

23. $11x^3y^3\sqrt{2y}$

24. $\dfrac{2x^3}{y}$

25. $\{-7, 4\}$

26. $\varnothing$

27. $\{1/4\}$

28. $\{2\}$

29. $-3 \pm 2\sqrt{5}$

30. $-9 \pm 4\sqrt{5}$

31. $2 \pm \sqrt{3}$

32. $\{-2.5, 3.5\}$

APPENDIX A

To Instructors

1. Lecture Notes

1.1. Concepts.

1. Expressions

Define: expressions, equivalent expressions, equations, relations, substitution, sum of terms, product of factors, coefficient. Show how to split sums into terms, and products into factors.

2. Axioms

Stop before identity to show that terms of sums can be added in any order, and factors of a product can be multiplied in any order. Stop before distributivity to discuss opposites and reciprocals. Show distributivity both ways.

EXAMPLE 1.1.1. Rewrite the expression without parentheses and then simplify by appealing to the axioms: $x + 3(2 + x)$

SOLUTION:

equivalent expressions	**axiom used for substitution**
$x + 3(2 + x)$	
$x + (3 \cdot 2 + 3 \cdot x)$	distributivity
$x + (3 \cdot x + 3 \cdot 2)$	commutativity of $+$
$x + (3 \cdot x + 6)$	closure
$(x + 3 \cdot x) + 6$	associativity of $+$
$(1 \cdot x + 3 \cdot x) + 6$	identity of $\cdot$
$(1 + 3)x + 6$	distributivity
$4x + 6$	closure

Define subtraction and division, show sums with negative terms and products with reciprocal factors.

3. Sets

Define sets, roster, integers: positive and non-negative.

Show the number line and the addition/subtraction of negative numbers.

Define set builder, show odd numbers or something like that:

$$\left\{2k+1 \,\middle|\, k \text{ is an integer}\right\}$$

Show how to list a few members of a set

$$\left\{m^2 \,\middle|\, m \text{ is a positive integer}\right\}$$

Define rationals and reals, mention irrational numbers.

Show absolute value and inequality relations.

4. Reals

Show products of negative numbers, opposite of a sum, positive integer exponent.

5. Order of Operations

Order of operations, combining like terms.

6. Primes

Define primes, show prime factorizations

Show cancellation of common factors in fractions, define fractions in lowest terms.

7. Fractions

Show reciprocal of a product, rewriting a fraction as a product, multiplying fractions with invisible parentheses, applications to areas, dividing fractions, and changing the denominator.

8. LCD and Fraction Addition

Show distributivity with common denominator, define LCM and LCD.

Show rational addition and mixed numbers.

9. Translation

Show the four arithmetic patterns, mention combinations like “X is 10 units greater than 4 times y”.

1.2. Linear Equations.

1. Properties

Define solution, equivalent equations, show addition and multiplication properties.

EXAMPLE 1.2.1. Ivan's math class has 1.2 times more students than Ivan's statistics class. Find how many people are enrolled in the statistics class has if the math class has 30 students.

SOLUTION: Let m and s be the enrollment figures for math and stat class respectively (units are students). The statement

math class has 1.2 times more students than statistics class

can be translated as

$$m = 1.2s$$

Since we are given that the math class has 30 students, we can substitute 30 for m in the equation and solve for s:

$$\begin{aligned} 30 &= 1.2s \\ 25 &= s \end{aligned}$$

So the stats class has 25 students enrolled.

ANSWER:
m and s count students enrolled in math and stats class respectively,
equation: $m = 1.2s$
solution: $s = 25$

2. Solving

Define linear equations. Show how to solve by isolating the variable: a simple case, with parentheses, and with fractions.

EXAMPLE 1.2.2. A bag of dog food is $7 more expensive than a bag of cat food. Megan pays $46 for two bags of cat food and one bag of dog food. Find the unit price for each item.

EXAMPLE 1.2.3. An ornamental rug is shaped like a rectangle, with one side 2.5 feet shorter than the other side. Find the dimensions of the rug if its perimeter is 33 feet.

EXAMPLE 1.2.4. Bart, Millhouse, and Martin Prince saved up 149 dollars for a collectible comic book. Find how much money each one saved up if Bart contributed 3 times less than Millhouse, and Martin contributed 9 dollars more than Millhouse.

SOLUTION: Let b, m, and p be the amounts contributed by Bart, Millhouse, and Martin Prince, in dollars. We can translate the statements as

$$\begin{aligned} b+m+p &= 149 && \text{together they saved up \$149} \\ b &= m \div 3 \\ p &= m+9 \end{aligned}$$

We can substitute expressions for b and p into the first equation and then solve for m:

$$\begin{aligned} b+m+p &= 149 \\ (m \div 3)+m+(m+9) &= 149 \\ m+3m+3m+27 &= 447 \\ 7m &= 420 \\ m &= 60 \end{aligned}$$

We can find b and p now using the other two equations which are solved for these variables:

$$\begin{aligned} b &= m \div 3 = 60 \div 3 = 20 \\ p &= m+9 = 60+9 = 69 \end{aligned}$$

ANSWER:
b, m, and p be the amounts contributed by Bart, Millhouse, and Martin Prince, in dollars,

$$\begin{cases} b+m+p=149 \\ b=m \div 3 \\ p=m+9 \end{cases}$$

solution: $b=20,\ m=60,\ p=69$

3. Formulas

Show how to solve linear formulas for a variable. Examples to be solved for y:

$$\begin{aligned} 3x-4y &= x-4 \\ 2(x-y) &= 4y+10x-1 \\ \frac{7xy}{3} &= \frac{4}{b} \\ 8y+2 &= 4(x-xy) \end{aligned}$$

4. Percent

Define percent, show decimal representation, show three easy translation patterns. Define percent increase and decrease, show basic examples and applications.

EXAMPLE 1.2.5. A quantity increased by 14% and is now 171. Find the quantity before the increase.

SOLUTION:

$$\begin{aligned} 171 &= P(1+0.14) \\ 150 &= P \end{aligned}$$

EXAMPLE 1.2.6. A quantity decreased from 140 to 133. Find the percent decrease.

SOLUTION:

$$\begin{aligned} 133 &= 140 - \frac{k}{100} \cdot 140 \\ -7 &= -\frac{k}{100} \cdot 140 \\ \frac{7 \cdot 100}{140} &= k \\ 5 &= k \end{aligned}$$

ANSWER: 5%

EXAMPLE 1.2.7. John paid 24% income tax on the total income of \$28,000. Find how much money John has made after tax, and the amount he paid to the government.

ANSWER: John has made \$21,280 after paying \$6,720 tax.

EXAMPLE 1.2.8. A contractor buys a residential property, renovates it, and then sells (flips) it with 25% markup. If the the sale price is \$160,000, find how much the contractor has paid for the property, as well as his profit from flipping it.

ANSWER: The original price was \$128,000, and the profit was \$32,000.

5. Applications

Show the work equation and give few basic examples of assigning/comparing units:

- distance = speed · duration
- paycheck = hourly rate · duration
- volume of water = pumping rate · duration
- total price = unit price · quantity

EXAMPLE 1.2.9. Ivan presented 9 examples over 2 hours and 15 minutes. Describe variables and units in

$$w = rt$$

and find the rate.

SOLUTION: w is the number of examples presented, t is duration in hours, r is the rate in examples per hour. Note that 15 minutes is 0.25 of an hour.

$$\begin{aligned} 9 &= r \cdot 2.25 \\ 4 &= r \end{aligned}$$

So the rate is 4 examples per hour.

EXAMPLE 1.2.10. Shannon's business serves 14 clients per day on average. Describe variables and units in

$$w = rt$$

and find the time needed to serve 518 clients?

SOLUTION: The rate is in clients per day, so the measure of "work" is just the number of clients, and the units of time are days.

$$\begin{aligned} 518 &= 14t \\ 37 &= t \qquad \text{days} \end{aligned}$$

EXAMPLE 1.2.11. Two cars, a Lotus and a Renault, are 635 kilometers apart. They start at the same time and drive toward each other. The Lotus travels at a rate of 70 kilometers per hour and the Renault travels at 57 kilometers per hour. In how many hours will the two cars meet?

SOLUTION:

$$\begin{aligned} 70t + 57t &= 635 && \text{distance equation} \\ 127t &= 635 \\ t &= 5 && \text{hours} \end{aligned}$$

EXAMPLE 1.2.12. Jane's monthly car loan payment is one quarter of her monthly house loan payment. Find each monthly payment rate if Jane pays 19200 dollars in total over the course of the year.

SOLUTION: c and h are payment rates for the car and the house respectively, in dollars per month.

$$\begin{aligned} c &= h \div 4 && \text{car payment is a quarter of house payment} \\ c \cdot 12 + h \cdot 12 &= 19200 && \text{“work” equation} \end{aligned}$$

Solve by substitution:

$$\begin{aligned} 12(h \div 4) + 12h &= 19200 && \text{substitute } h/4 \text{ for } c \\ 3h + 12h &= 19200 \\ 15h &= 19200 \\ h &= 1280 && \text{house payment} \end{aligned}$$

Back to the first equation to find c:

$$\begin{aligned} c &= 1280/4 \\ c &= 320 && \text{car payment} \end{aligned}$$

EXAMPLE 1.2.13. Wayne and Dennis like to ride the bike path from Riverside Park to the beach. Dennis' speed is seven miles per hour faster than Wayne's speed, so it takes Wayne two hours to ride to the beach while it takes Dennis one hour and thirty minutes for the ride. Find the speed of each biker.

6. Inequalities

Define linear inequalities and solutions. Show graphs and interval notation. Show addition and multiplication properties, examples.

1.3. Graphing.

1. Reading

Define ordered pair, coordinate plane, graph of a set of points. Plot/read some points, show axis scaling, show how to find intercepts and extrema.

2. Linear Equations

Define solution set. Show some fun solution sets, like $y = x^2$ or $y = x^3$. Show vertical and horizontal lines. Show how to graph linear equations which can be solved for a variable by finding 2 points.

3. Intercepts

Define intercepts, show how to find them from equations, and how to use them for plotting. Mention vertical and horizontal lines.

4. Slope

Define grade, slope of a segment, slope of a line. Mention positive and negative slopes. Show how to plot using a rational slope.

5. Slope-intercept

Define slope-intercept form. Show how to read, write, and to solve for y to get it. Define parallel and perpendicular lines, show how to compare given lines using their slopes, mention vertical and horizontal lines.

6. Point-slope

Define point-slope form, show how to read. Show how to construct an equation of a line given slope and point, and also given two points. Show how to use point-slope form for plotting on the grid.

7. Function Notation

This section is optional in the sense that nothing else in the text depends on it.

Define functions and function notation, show examples of use.

EXAMPLE 1.3.1. Acme CO's profit in 2000 is 4.7 billion dollars, and 5.3 billion dollars in 2015. Let x be the year since 2000, and let $y(x)$ be the profit in the corresponding year, in billions of dollars. Find a linear model $y(x)$, use the model to predict the profit in 2018, and to predict the year when the profit is 6 billion dollars.

SOLUTION:

$$\begin{aligned}
m &= 0.6/15 = 0.04 && \text{slope}\\
y(x) &= 0.04x + 4.7 && \text{model}\\
y(18) &= 5.42 && \text{billion dollars}\\
6 &= 0.04x + 4.7 &&\\
1.3 &= 0.04x &&\\
32.5 &= x && \text{approximately year 2032}
\end{aligned}$$

1.4. Systems.

1. Graphing

Define the solution for a system with 2 equations and 2 variables. Show the 3 cases for the shape of the solution set. Show how to solve by graphing:

$$\begin{cases} 2x + 4y = 4 \\ 2x - y = 9 \end{cases} \qquad (4, -1)$$

$$\begin{cases} y + x = 3 \\ 2y = 7 - 2x \end{cases} \qquad \text{no solution}$$

2. Substitution

Solve by substitution:

$$\begin{cases} y = 5x + 13 \\ 4x - 3y = -17 \end{cases} \qquad (-2, 3)$$

$$\begin{cases} 3y - 2x = -4 \\ 4x + 5y = -36 \end{cases} \qquad (-4, -4)$$

3. Elimination

Show multiplication and addition properties for systems. Solve by elimination:

$$\begin{cases} 5x - 6y = 11 \\ 5x + 6y = 59 \end{cases} \qquad (7, 4)$$

$$\begin{cases} 3x - 7y = 11 \\ 9x + 5y = -19 \end{cases} \qquad (-1, -2)$$

4. Applications

EXAMPLE 1.4.1. Every Monday Helen shares coffee and donuts with the office where she works. One week Helen buys 4 cups of coffee and 6 donuts, and pays 18 dollars. The next week Helen buys 6 cups of coffee and 12 donuts, and pays 30 dollars. Find the price of one cup of coffee and the price of one donut.

SOLUTION: Let c and d be the price of one cup of coffee and the price of one donut respectively, in dollars.

$$\begin{cases} 4c + 6d &= 18 \\ 6c + 12d &= 30 \end{cases}$$

ANSWER: $(3, 1)$

EXAMPLE 1.4.2. Morgan's drinkable water reserve consists of a number of large water jugs as well as some smaller plastic bottles for individual consumption. Each jug contains 7 liters of water, and each bottle contains 0.6 liters of water. Find how many bottles and how many jugs of water Morgan has, if the total volume of the stored water is 158 liters, and there are 50 water containers in total.

EXAMPLE 1.4.3 (Volume Mixing). A chemist needs to mix an 18% acid solution with a 45% acid solution to obtain 12 liters of a 36% solution. How many liters of each of the acid solutions must be used?

SOLUTION: Let x and y be amounts of solutions in liters.

$$x + y = 12$$
$$0.18x + 0.45y = 0.36(12)$$

EXAMPLE 1.4.4 (Value Mixing). How many pounds of chamomile tea that cost \$9 per pound must be mixed with 8 pounds of orange tea that cost \$6 per pound to make a mixture that costs \$7.80 per pound?

SOLUTION: Let c be the weight of chamomile tea in pounds, and then

$$9c + 6(8) = 7.8(c + 8)$$

Alternatively, we can also let t be the total weight of the mix, and write a system of two equations:

$$c + 8 = t$$
$$9c + 6(8) = 7.8t$$

5. Inequalities

Define the solution for an inequality in 2 variables, describe solution sets, show examples, mention vertical and horizontal lines.

1.5. Polynomials.

1. Exponent

Define integer exponent, mention base 1. Show product rule and quotient rule. Mention sums as bases: $(x + y)^n$

2. Properties

Show power rule and distributivity over products and fractions.

3. Polynomials

Define monomial and its simplified form, monomial degree, monomial degree names, polynomial and its standard form, leading coefficient, polynomial names, polynomial degree. Show how to combine like terms and how to evaluate polynomials.

4. Sums

Show how to add and subtract polynomials by combining like terms.

5. Products

Show how to multiply monomials, monomial times polynomial, and polynomial times polynomial (show the box).

6. Special Products

Show the difference of squares both ways, and the square of a binomial.

7. Quotients

Show how to divide by a monomial, and the polynomial long division.

8. Negative Exponent

Mention definition again, show how to get rid of negative exponents in products and fractions. Define scientific notation.

1.6. Factoring.

1. GCF

Define irreducible and prime polynomials. Show polynomial factorizations. Mention that linear polynomials are irreducible.

Define GCF for integers and GCF for monomials.

2. Grouping

Show factoring by grouping.

$$xy+2y+3x+6 \qquad 4x^3+2x^2+2x+1$$
$$24x^3-6x^2+8x-2 \qquad 3x^3-6x^2-15x+30$$

3. Diamond

Show the diamond pattern. Show the basic factoring strategy combining monomial GCF, grouping, and diamond.

4. Trinomials

Show how to factor a trinomial.

$$6z^2+11z+4 \qquad -4h^2+11h+3$$
$$2n^2-13n-7 \qquad 3x^2+xy-14y^2$$
$$4r^2+11rs-3s^2 \qquad 5x^2+14x-3$$

5. Special Products

Show how to factor special products.

6. Strategy

List the types of irreducible polynomials: linear, prime trinomials, and sums of squares. Present the factoring strategy, combining GCF, grouping, difference of squares, diamond, and trinomial pattern.

$$6x^4-13x^3+5x^2 \qquad 5x^5-20x$$
$$4x^2y-9y+8x^2-18 \qquad 36x^4-15x^3-9x^2$$

7. Equations

Show the zero product property and how to solve non-linear polynomial equations.

$$\begin{aligned} 3x^2 - 12x &= 0 \\ x^3 + 5x^2 - 9x - 45 &= 0 \\ 3x^5 - 6x^4 + 3x^3 &= 0 \\ 6x^3 - 4x^2 - 16x &= 0 \\ 2x^3 + 10x - 3x^2 - 15 &= 0 \end{aligned}$$

8. Applications

Show applications to areas of rectangles and triangles.

1.7. Rational Expressions.

1. Simplifying

Define rational expressions. Show how to find which values of a variable make the expressions undefined. Show how to simplify by canceling common polynomial factors.

$$\frac{6x^3 - 12x^2 - 18x}{3x^4 + 6x^3 + 3x^2}$$

2. Products

Show how to multiply and divide rational expressions.

$$\frac{30x^2 - 15x}{6x - 42} \cdot \frac{x^2 - 49}{20x^2 - 20x + 5} \qquad \frac{x - 4}{x^2 - 9} \div \frac{x^2 - x - 12}{x^2 + 6x + 9}$$

3. LCD

Define common multiple, LCM, and LCD. Show how to find LCMs and LCDs.

4. Sums

Show how to simplify sums with the same denominator, and sums with different denominators.

$$\frac{x^2}{x - 5} - \frac{25}{x - 5} \qquad \frac{2}{t + 1} - \frac{t - 2}{t^2 - t - 2}$$

$$\frac{x + 3}{x^2 + 6x + 9} - \frac{x - 3}{2x^2 + 6x} \qquad \frac{2x^2}{x^2 - 4} + \frac{x}{x - 2} - \frac{1}{x + 2}$$

5. Complex Fractions

Define complex fractions, show how to simplify them.

$$\frac{3+\frac{9}{x-3}}{4+\frac{12}{x-3}} \qquad \frac{\frac{y}{x}-\frac{x}{y}}{\frac{1}{y}+\frac{1}{x}} \qquad \frac{\frac{1}{x}+4}{3+\frac{2}{x^2}}$$

6. Equations

Define extraneous solutions. Show how to detect extraneous solutions for equations involving rational expressions. Show how to solve the equations.

$$\begin{aligned}
\frac{6}{x-3}+9 &= \frac{-3}{x-3}\\
\frac{3}{x^2-2x} &= \frac{1}{x-2}-\frac{2}{x^2}\\
\frac{x}{x+2}+\frac{2}{x^2+5x+6} &= \frac{5}{x+3}\\
\frac{x}{x-5} &= \frac{7x}{x^2-3x-10}-\frac{3}{x+2}\\
\frac{5}{n^3+5n^2} &= \frac{4}{n+5}+\frac{1}{n^2}
\end{aligned}$$

1.8. Radicals.

1. Intro

Define principal square root. Mention that the square root of a negative number is not real, but complex. Show how the square root cancels the even exponent. Define principal nth root and show how it cancels the exponent which is multiple of n. Mention sums as radicands.

2. Products

Show distributivity over multiplication. Use it to simplify the radicand involving numbers as well as variables:

$$\sqrt{24} \qquad \sqrt{a^7}$$

$$\sqrt{5x}\cdot\sqrt{5x^3} \qquad \sqrt{6}(2\sqrt{3}+3\sqrt{6})$$

3. Quotients

Show distributivity over fractions. Define rationalized denominator. Show how to simplify fractions and rewrite them with rationalized denominator:

$$\frac{1}{\sqrt{5}} \qquad \frac{\sqrt{8}}{\sqrt{2y}} \qquad \frac{\sqrt{40x}}{\sqrt{2x^2}}$$

4. Sums

Define radical like terms. Show how to simplify sums by simplifying radicals and combining like terms:

$$\sqrt{3}+\sqrt{27} \qquad\qquad (3-\sqrt{2})(5-2\sqrt{2})$$

Define radical conjugates $A\sqrt{B}\pm C\sqrt{D}$. Show how to rationalize a denominator with a sum of two terms.

5. Equations

Show that squaring both sides produces an almost equivalent equation. Show how to check for extraneous solutions:

$$\sqrt{2x+1}=x-1, \quad \text{check } x=4 \text{ and } x=0$$

Show how to solve equations:

$$\begin{aligned}
7-\sqrt{3x+7} &= 3\\
2+6\sqrt{2x^2-3x+5} &= 0\\
4+\sqrt{x^2-4x} &= x\\
\sqrt{x+\sqrt{x-1}+2} &= 3\\
\sqrt{3x-5} &= 2-\sqrt{x-1}
\end{aligned}$$

6. Applications

Show the distance formula and applications to right triangles.

7. Rational Exponent

Define rational exponent and show why it has to be in lowest terms before it can be evaluated. Show how to change expressions from radical to rational exponent form and back. Show how to evaluate expressions. Show how to simplify expressions involving rational exponents. Optionally, define the imaginary unit and show how the distributivity and the exponent of exponent property fail for negative bases.

1.9. Quadratics. It is possible to skip introducing complex numbers, if so desired, by skipping the first section. The homework in the sections that follow is divided into two parts: first part with real solutions, and the second part with complex solutions, so one can easily assign one or the other or both topics to practice.

1. Complex Numbers

Simplify square roots of negative numbers, sums and products of complex numbers, and expressions of the form $\dfrac{-b\pm\sqrt{D}}{2a}$

2. Square Root Property

State that every $w^2 = z$ has two solutions for non-zero z. Show the square root property and the plus-minus notation. Show how to solve equations $(nx + k)^2 = r$.

3. Completing The Square

Show how to complete the square $x^2 + bx$ as well as $ax^2 + bx$. Show how to solve equations using this method.

4. Quadratic Formula

Derive the quadratic formula by completing the square in

$$ax^2 + bx + c = 0$$

Show a few examples.

APPENDIX B

To Developers

1. Contributing

1.1. Submitting Contributions. If you would like to suggest an edit, please do not hesitate to contact the maintainer:

"Ivan G. Zaigralin" <melikamp@melikamp.com>

Both this text and its source are licensed under free and libre licenses of the copyleft type, meaning that anyone can use them in any way whatever, up to and including distributing modified copies, as long as they use the same licenses for derivative works. All your contributions automatically assume the same project license, but nontrivial LaTeX source copy may qualify you for becoming a co-author and a consequent copyright assignment.

Awesome ideas about content, structure, and appearance are welcome, and so is constructive criticism. If you want more exercises or examples of a certain type, we urge you strongly to do as much work as possible, and describe the statements, the solutions, and the answers in detail, so that we can typeset your idea with minimal research. And if you happen to know LaTeX and opt to submit code contributions which are formatted as described below, you should be prepared to be listed as a co-author.

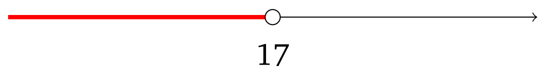
1.2. Document Hierarchy. Standard book template hierarchy is used, with chapters, sections, and subsections. The chapter source files are included from `basic-algebra.tex`. Each chapter file contains the title and the list of included sections. Each section source lives in its own file. When working on a specific section, it is possible to greatly speed up the build time by commenting out other chapters in `basic-algebra.tex`, and other sections in the corresponding chapter source file. The chapter title is printed on the same page as the first section in that chapter, so do not forget to comment out the title as well when working on sections other than the first, or else the document will re-flow incorrectly.

2. Document Formatting Guidelines

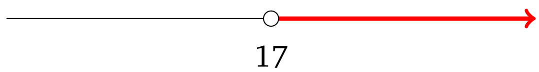
2.1. Mathematical Notions. We use fairly vanilla LaTeX throughout the text, only defining special commands when we really need to enforce the way a particular concept is typeset.

- Interval graphs

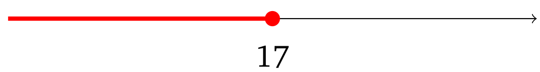
  `\InfOpen{17}`

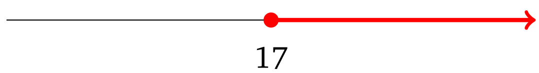
  `\OpenInf{17}`

  `\InfClosed{17}`

  `\ClosedInf{17}`

- Interval notation

  `\opop{1,2}`

  $(1,2)$

  `\opcl{1,2}`

  $(1,2]$

  `\clop{1,2}`

  $[1,2)$

  `\clcl{1,2}`

  $[1,2]$

- Large delimiters

  `\pipes{x+\sqrt x}`

  $\left|x+\sqrt{x}\right|$

  `\brackets{x+y}`

  $\left[x+y\right]$

  `\parens{\frac1x + 3}`

  $\left(\frac{1}{x}+3\right)$

- Mixed number

  `\mixed{3}{4}{5}`

  $3\frac{4}{5}$

- Set builder

  `\setb{2k}{k\in A}`

  $\left\{2k \,\middle|\, k \in A\right\}$

- Set roster

  `\aggr{1,2,3}`

  $\{1,2,3\}$

- Complex fractions set in normal font size

  `\compfrac{\frac 12 + \frac 34}{\frac1x - \frac2x}`

  $$\frac{\dfrac{1}{2}+\dfrac{3}{4}}{\dfrac{1}{x}-\dfrac{2}{x}}$$

- System of two equations

  `\twoEqSystem{x + y}{2x + 2}{y}{4x - 5}`

  $$\begin{cases} x+y &= 2x+2 \\ y &= 4x-5 \end{cases}$$

2.2. Sets of Equations and Expressions. Equivalent expressions and relations are typeset using `align`, with commands defined to make the relation appear nice:

```
\begin{align*}
  x - 1 &\ee 5 &\ecomment{ecomment is for annotations in equations}\\
  x &\ee 6 &\ecomment{ee command makes equal sign with space around it}
\end{align*}
```

$$\begin{aligned} x-1 &= 5 && \text{ecomment is for annotations in equations} \\ x &= 6 && \text{ee command makes equal sign with space around it} \end{aligned}$$

```
\begin{align*}
  x - 1 &\qq\geq 5\\
  x &\qq\geq 6 &\ecomment{qq command inserts space around the argument}
\end{align*}
```

$$\begin{aligned} x-1 &\geq 5 \\ x &\geq 6 && \text{qq command inserts space around the argument} \end{aligned}$$

2.3. Frames. There are several `mdframed` environments defined, and should be used consistently throughout.

```
\begin{axiom}[This is an axiom]
  This environment is used for the real and complex number axioms only.
\end{axiom}
```

AXIOM 2.3.1 (This is an axiom). This environment is used for the real and complex number axioms only.

```
\begin{definition}[This is a definition]
  This environment is used for definitions.
\end{definition}
```

DEFINITION 2.3.1 (This is a definition). This environment is used for definitions.

```
\begin{theorem}[This is a true fact]
  This environment is used for all general enough true statements
  which can be proven from the axioms and the logical principles.
\end{definition}
```

THEOREM 2.3.1 (This is a true fact). This environment is used for all general enough true statements which can be proven from the axioms and the logical principles.

```
\begin{trivia}[This is an easy example]
  This environment is used for examples which are trivial enough.
\end{trivia}
```

BASIC EXAMPLE 2.3.1 (This is an easy example). This environment is used for examples which are trivial enough.

```
\begin{example}[This is a nontrivial example]
  This environment is used for stating examples which are similar to
  the homework. It is almost always followed by the other two
  environments: the solution and the answer.
\end{example}
\begin{exsol}
  This environment provides a solution for the example stated above.
\end{exsol}
\begin{exans}
  This environment states the answer to the question above in an
  optimal way which the readers are expected to follow as they are
  working through the homework.
\end{exans}
```

EXAMPLE 2.3.1 (This is a nontrivial example). This environment is used for stating examples which are similar to the homework. It is almost always followed by the other two environments: the solution and the answer.

SOLUTION: This environment provides a solution for the example stated above.

ANSWER: This environment states the answer to the question above in an optimal way which the readers are expected to follow as they are working through the homework.

2.4. Homework and Answers.

```
\begin{hw-and-answers}
  \begin{exercise}
    These three environments allow to typeset exercise sections with
    or without answers. They are numbered as subsections and they insert
    page breaks. Does this exercise have an answer?
    \begin{answer}
      Yes.
    \end{answer}
  \end{exercise}

  \begin{exercise}
    This exercise does not have an answer. Why should it?
  \end{exercise}
\end{hw-and-answers}
```

Homework B.2.

1. These three environments allow to typeset exercise sections with or without answers. They are numbered as subsections and they insert page breaks. Does this exercise have an answer?

2. This exercise does not have an answer. Why should it?

Homework B.2 Answers.

1. Yes.

3. To Do

3.1. To Do Items.

- More notes for applications of $w = rt$.
- More applications of difference in chapter 1.
- More applications of slope as rate of change.
- More mixture notes.
- Add 1-variable mixture problems in chapter 2.
- Add applications of equations with rational expressions?
- Add practice test 9?

4. Working with the Source

4.1. Getting the Source. The easiest way to get the source is to `git` it. You can clone the repository with

```
git clone https://git.albertleadata.org/melikamp/basic-algebra
```

4.2. Building the Source with LaTeX. This text is typeset in LaTeX, texlive to be specific, and a GNU+Linux environment which provides a minimal userland such as `ls` and a `bash`-like shell, a `pdflatex` command and GNU `make`. If you use any kind of vanilla GNU+Linux OS such as Freenix or Debian, then all you need to do is fire up the console, change into the directory with the source, and run

```
make
```

to build `print/basic-algebra.pdf`, which is the version intended as the final result.

```
make clean
```

will remove temporary build files, staged in `build/`.

If you happen to use a bizarro OS without a GNU-style `make` and/or `texlive`, you may still be able to build the PDF. At the very least you need some kind of `pdflatex` and `makeindex` commands, and all the LaTeX packages used. Then you just need to run something like

```
pdflatex basic-algebra

pdflatex basic-algebra

makeindex basic-algebra

pdflatex basic-algebra

pdflatex basic-algebra
```

to build the final version of the PDF in the current directory. Good luck.

4.3. Branches. The development happens in the branch numbered after the next edition (**first**, **second**, and so on). Once the new edition is ready for use, the changes are merged into the **master** branch, and a corresponding tag is created. The top of the **master** branch is then merged into the **color** and **mono** branches. The **master** branch, while printable, is intended to be viewed on a screen; **color** is a print-ready branch with decolored web-links, and **mono** is a print-ready branch done mostly in shades of gray, with graphs optimized for mono printers.

5. Acknowledgments

Many homework exercises were shamelessly borrowed from *Beginning and Intermediate Algebra*, a free and libre (CC-BY) textbook by Tyler Wallace, available for free download at http://wallace.ccfaculty.org/book/book.html

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